MAT334 - Complex Variables

Contents

1 | The Complex Numbers

This text is entirely based around the study of $\mathbb{C}$, the set of complex numbers. How do complex functions behave? Can we differentiate and integrate? How does this differ from working over $\mathbb{R}$? In due time, we will see all of this. But before we can get to any of that, we need to discuss what $\mathbb{C}$ is and how to do algebra in the complex numbers.

1.1 What is $\mathbb{C}$?

Definition 1.1.1: The Complex Numbers

The imaginary unit i is a number such that $i^2 = -1$. A complex number is a number of the form $a + bi$, where $a, b \in \mathbb{R}$. The set of complex numbers is $\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$.

Note. The name "imaginary" is a misnomer. When mathematicians first started thinking about complex numbers, i was treated at best as a calculation trick and at worst as pure nonsense. The name was originally chosen as an insult.

Now, we recognize that the concept is real in the same way any other advanced mathematics is. These actually exist, and we can formally design a system that exhibits this behavior.

Beyond that, however, complex numbers are actually useful in real life. For example, the mathematics behind electomagnetism is based on working with complex numbers.

Example 1.1.1

For example, $2 + 3i$, $\pi - ie^2$, and 1 are all complex numbers.

Why would 1 be a complex number? Isn't it real? When we write 1 in this context, we mean $1 + 0i$. In this way, we can think of every real number r as a complex number as well: $r = r + 0i$.

1.2 Complex Algebra

Let's talk about how to manipulate complex numbers. Our overarching goal is to develop some notion of calculus. This requires us to be able to do algebra on $\mathbb{C}$.

Definition 1.2.1: Real and Imaginary Parts

Let $a + bi \in \mathbb{C}$. Then the **real and imaginary parts** of $a + bi$ are:

$$\mathrm{Re}(a + bi) = a$$
$$\mathrm{Im}(a + bi) = b$$

Example 1.2.1

Consider $z = 3 + 4i$. The real part of z is 3, and the imaginary part is 4. Notice: 4, not $4i$. The imaginary part of z is still a real number.

Definition 1.2.2: Addition

Let $a + bi, c + di \in \mathbb{C}$. Then:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

So adding $z, w \in \mathbb{C}$ is done by adding together the real parts of z, w, and adding together the imaginary parts.

Definition 1.2.3: Multiplication

Let $a + bi, c + di \in \mathbb{C}$. Then:

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

Why would we choose this definition? Well, we want complex multiplication to satisfy the "distributivity property": $(a + b)c = ac + bc$ and $a(b + c) = ab + ac$. If we require these to hold, then we are forced to conclude that:

$$\begin{aligned}
(a + bi)(c + di) &= (a + bi)c + (a + bi)di \\
&= (ac + bic) + (adi + bidi) \\
&= ac + bci + adi + bdi^2 \\
&= ac + bci + adi - bd \\
&= (ac - bd) + (ad + bc)i
\end{aligned}$$

Complex multiplication and addition satisfy a whole bunch of properties, specifically what are called the field axioms.

Theorem 1.2.1: The Field Axioms

The complex numbers satisfy the following properties. For all $u, w, z \in \mathbb{C}$:

1. $w + z = z + w$

2. $u + (w + z) = (u + w) + z$
3. $z + 0 = z$
4. If $z = x + iy$, then $-z = (-x) + i(-y)$ satisfies that $z + (-z) = 0$.
5. $wz = zw$
6. $u(wz) = (uw)z$
7. $1z = z$
8. For any $z \in \mathbb{C}$ with $z \neq 0 + 0i$, there exists some $w \in \mathbb{C}$ with $zw = 1$.
9. $u(w + z) = uw + uz$ and $(u + w)z = uz + wz$

Proof. Many of these properties aren't hard to check, and follow pretty quickly from facts you know about real numbers.

We will provide a proof for the existence of multiplicative inverses very shortly, before we discuss division. □

Example 1.2.2

Let $z = 2 + 7i$, $w = 4 - 3i$. Find $w^2 - zw$.

Well, to make life simpler, we can factor (using distributivity):

$$w^2 - zw = w(w - z) = (4 - 3i)[(4 - 3i) - (2 + 7i)] = (4 - 3i)(2 - 10i) = (8 - 30) + (-40 - 6)i = -22 - 46i$$

What about division? First, what is division? What does $\frac{1}{z}$ mean?

Definition 1.2.4: Multiplicative Inverse

Let $z \in \mathbb{C}$. We say that $w = \frac{1}{z}$ if $zw = 1$. In this situation, w is a **multiplicative inverse** for z. The existence of such a w is one of the field axioms.

So how do we find a multiplicative inverse for z? To do that, we're going to need to introduce two new ideas, the complex conjugate and the modulus:

Definition 1.2.5: Complex Conjugate

Let $a + bi \in \mathbb{C}$. Then the **complex conjugate** of $a + bi$ is:

$$\overline{a + bi} = a - bi$$

Definition 1.2.6: Modulus

Let $a + bi \in \mathbb{C}$. Then the **modulus** of $a + bi$ is the real number:

$$|a + bi| = \sqrt{a^2 + b^2}$$

How does this help us define division? It turns out, these are exactly the building blocks we need:

Lemma 1.2.1. *Let $a+bi \in \mathbb{C}$ be non-zero (i.e., at least one of a, b is not 0). Then $\frac{1}{z} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$. Another way of writing this is that $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$, where we understand that division by a real number means division on both the real part and imaginary part.*

Proof. To show that $\frac{1}{z}$ is what we claim, we need only check that $z\frac{1}{z} = 1$. We compute:

$$\begin{aligned}(a + bi)\left(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i\right) &= \frac{a^2 - (b)(-b)}{a^2 + b^2} + \frac{ab + a(-b)}{a^2 + b^2}i \\ &= \frac{a^2 + b^2}{a^2 + b^2} \\ &= 1\end{aligned}$$

□

Where did this come from? The intuition comes from the calculation:

$$\frac{1}{a + bi}\frac{a - bi}{a - bi} = \frac{a - bi}{(a - bi)(a + bi)} = \frac{a - bi}{a^2 + b^2}$$

We can't use this as a proof of the result, since it relies on being able to divide by complex numbers (which we haven't even defined yet!), as well as being able to manipulate fractions. However, as far as intuition goes, this is a good way to understand the result.

Note. This calculation also shows that $z\overline{z} = |z|^2$. This is a very important fact, and will come up very frequently.

This lemma lets us define division properly:

Definition 1.2.7: Division

Let $z, w \in \mathbb{C}$ with $w \neq 0$. Then:

$$\frac{z}{w} = \frac{z\overline{w}}{|w|^2}$$

Example 1.2.3

Let $z = 4 + 3i$ and $w = 1 + i$. Find $\frac{z}{w}$ and $\frac{w}{z}$.

We know from our formula that $\frac{z}{w} = \frac{z\overline{w}}{|w|^2}$. So we compute:

$$\overline{w} = 1 - i$$

$$|w|^2 = (1^2 + 1^2) = 2$$

So, $\frac{z}{w} = \frac{(4+3i)(1-i)}{2} = \frac{7}{2} - \frac{1}{2}i$.

To compute $\frac{w}{z}$, we could go through the same process. Or, we could note that $\frac{w}{z} = \frac{1}{\left(\frac{z}{w}\right)} = \frac{2}{7-i}$. We then get that:

$$\frac{w}{z} = \frac{2(7+i)}{|7-i|^2} = \frac{14}{50} + \frac{2}{50}i$$

How do complex conjugation and the modulus behave when combined with our algebraic operations?

Theorem 1.2.2: Algebraic Properties of Conjugation and the Modulus

Let $z, w \in \mathbb{C}$.

1. $\overline{zw} = \overline{z}\,\overline{w}$
2. $\overline{z+w} = \overline{z} + \overline{w}$
3. $\overline{\overline{z}} = z$.
4. $z + \overline{z} = 2\,\mathrm{Re}(z)$ and $z - \overline{z} = 2\,\mathrm{Im}(z)i$.
5. $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$
6. $|zw| = |z||w|$
7. $|\overline{z}| = |z|$
8. If $z = x + iy$, then $|x| \leq |z|$ and $|y| \leq |z|$.
9. The **triangle inequality**: $|z+w| \leq |z| + |w|$. And $|z+w| = |z| + |w|$ if and only if $z = rw$ for some $r \in \mathbb{R}$ or $w = rz$ for some $r \in \mathbb{R}$.
10. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

Proof. We will prove some, but not all of these statements. Many of them are variations on the same idea, and so will be quick to verify.

Let $z = a + bi$ and $w = c + di$.

1. From the definition, $zw = ac - bd + (ad + bc)i$. Therefore,

$$\overline{zw} = (ac - bd) - (ad + bc)i$$

On the other hand, we find that:

$$\overline{z}\,\overline{w} = (a - bi)(c - di) = (ac - (-b)(-d)) + (a(-d) + (-b)c)i = (ac - bd) - (ad + bc)i$$

So we see that $\overline{zw} = \overline{z}\,\overline{w}$.

4. We have $z + \overline{z} = a + bi + (a - bi) = 2a = 2\,\mathrm{Re}(z)$, and $z - \overline{z} = (a + bi) - (a - bi) = 2bi = 2\,\mathrm{Im}(z)i$.

8. Note that $|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2}$, since $f(x) = \sqrt{x}$ is an increasing function on $[0, \infty)$ and $y^2 \geq 0$. This shows that $|x| \leq |z|$. A similar argument shows that $|y| \leq |z|$.

9. To show this, it is enough to show that $|z + w|^2 \leq (|z| + |w|)^2$. It will take us a bit of work to get there though.

   First, notice that $(ad - bc)^2 \geq 0$. This implies that

$$2abcd \leq a^2d^2 + b^2c^2$$

   Adding $a^2c^2 + b^2d^2$ to both sides tells us that:

$$(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)$$

   As such, it follows that $2(ac + bd) \leq 2\sqrt{(a^2 + b^2)(c^2 + d^2)}$. Now, adding $a^2 + b^2 + c^2 + d^2$ to both sides gives:

$$(a^2 + 2ac + c^2) + (b^2 + 2bd + d^2) \leq (a^2 + b^2) + 2\sqrt{(a^2 + b^2)(c^2 + d^2)} + (c^2 + d^2)$$

   Factoring each side gives:

$$(a + c)^2 + (b + d)^2 \leq (\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2})^2$$

   However, $(a + c)^2 + (b + d)^2 = |z + w|^2$, $\sqrt{a^2 + b^2} = |z|$, and $\sqrt{c^2 + d^2} = |w|$. So we have shown that $|z + w|^2 \leq (|z| + |w|)^2$. Taking the square root of both sides gives $|z + w| \leq |z| + |w|$.

   For equality, note that if $z = rw$ or $w = rz$ (and you do need either condition, since you could have $z = 0$ or $w = 0$), then $ad - bc = 0$. In that case:

$$2abcd = a^2d^2 + b^2c^2$$

   And rather than having $\leq$ in our calculations above, equality follows through.

   Note. Fisher has a shorter proof of this fact in section 1.2. However, he does not discuss equality.

10. To begin, let's show that $\left|\frac{1}{w}\right| = \frac{1}{|w|}$. Notice that $\frac{1}{w} = \frac{c}{c^2+d^2} - \frac{d}{c^2+d^2}i$. And so:

$$\begin{aligned}
\left|\frac{1}{w}\right| &= \sqrt{\left(\frac{c^2}{(c^2+d^2)^2} + \frac{d^2}{(c^2+d^2)^2}\right)} \\
&= \sqrt{\frac{1}{c^2+d^2}} \\
&= \frac{1}{\sqrt{c^2+d^2}} \\
&= \frac{1}{|w|}
\end{aligned}$$

Now, by the first result of this theorem, we see that:

$$\left|\frac{z}{w}\right| = |z|\left|\frac{1}{w}\right| = |z|\frac{1}{|w|} = \frac{|z|}{|w|}$$

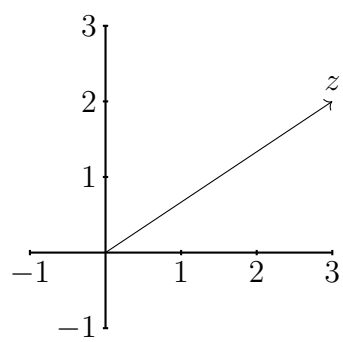
□

1.3 Rectangular Coordinates and $\mathbb{R}^2$

One way of visualizing complex numbers is to picture them as a vector in $\mathbb{R}^2$.

Example 1.3.1

Consider $z = 3 + 2i$. Then we can visualize z as the vector:



Note. In general, we can visualize the complex number $z = x + iy$ as the point (x, y) on the plane.

However, this does not mean that $\mathbb{C}$ and $\mathbb{R}^2$ are the same thing. They are not. We cannot multiply vectors in $\mathbb{R}^2$. We can't divide by vectors. Etc.

Definition 1.3.1: Rectangular Coordinates

The **rectangular coordinates** of $z \in \mathbb{C}$ is the point $(\mathrm{Re}(z), \mathrm{Im}(z))$ in $\mathbb{R}^2$.

When we write $z = x + iy$, we say that z is written in **rectangular form**.

This perspective does allow us to give some fairly nice proofs. For example:

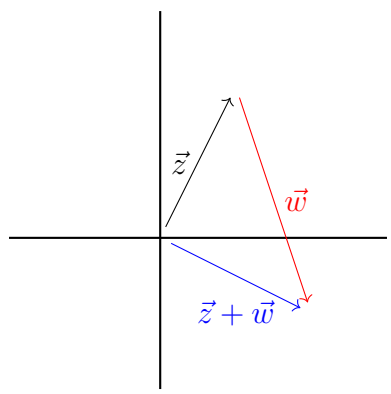
Theorem 1.3.1: The Triangle Inequality

Let $z = x + iy$. Then $|z|$ is equal to the length of the vector (x, y).

As a consequence, $|z + w| \leq |z| + |w|$.

Proof. The first claim is fairly straightforward: $|z| = \sqrt{x^2 + y^2}$, and the length of the vector (x, y) is $\|(x, y)\| = \sqrt{x^2 + y^2}$.

So, let $w, z \in \mathbb{C}$ and let $\vec{w}$ and $\vec{z}$ be their corresponding vectors in $\mathbb{R}^2$. Now, consider the following picture:



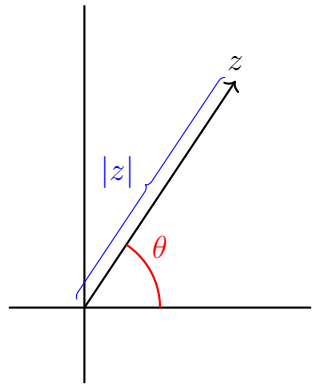
The vectors $\vec{z}$, $\vec{w}$, and $\vec{z} + \vec{w}$ satisfy the triangle inequality: $\|\vec{z} + \vec{w}\| \leq \|\vec{z}\| + \|\vec{w}\|$. And further, equality occurs precisely when $\vec{z}$ and $\vec{w}$ are in the same direction.

However, since the lengths of these vectors are equal to the corresponding moduli of the complex numbers, we conclude the triangle inequality on $\mathbb{C}$.

$\square$

1.4 Polar Coordinates

Let $z = x + iy$. We have seen that there is one way to represent complex numbers visually, via rectangular coordinates. There is another, which is in many ways much more important. Consider the picture:



We can also express this vector in $\mathbb{R}^2$ by giving its length, $|z|$, and the angle it forms with the positive x-axis, θ.

Lemma 1.4.1. *If $z \in \mathbb{C}$ with $|z| = r$, and z forms an angle of θ with the positive x-axis, then:*

$$z = r(\cos(\theta) + i\sin(\theta))$$

Proof. Suppose $|z| = r$. Then the vector $\vec{z}$ in $\mathbb{R}^2$ has $\|\vec{z}\| = r$. As such, $\vec{z}$ is on a circle of radius r, which we know (from MAT235) is parametrized by the equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$, where θ is the angle between $\vec{z}$ and the positive x-axis.

As such, we conclude that $z = x + iy = (r\cos(\theta)) + (r\sin(\theta))i$, as desired. □

Definition 1.4.1: Polar Coordinates

Let $z \in \mathbb{C}$. Then we say that z is in **polar coordinates**, or in **polar form**, if we write z as $z = r(\cos(\theta) + i\sin(\theta))$.

In this expression, $r = |z|$ and θ is the angle between z and the positive x-axis.

Example 1.4.1

Find the real and imaginary parts of $z = 2(\cos(0.2) + i\sin(0.2))$.

Well, $z = 2\cos(0.2) + (2\sin(0.2))i$, and so $\mathrm{Re}(z) = 2\cos(0.2)$ and $\mathrm{Im}(z) = 2\sin(0.2)$.

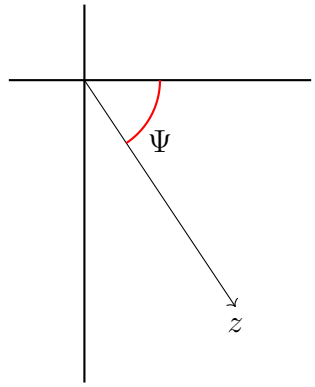
Note. Be careful! The real part of z is not 2. 2 is its modulus! This is a very common mistake!

Example 1.4.2

Write $|z| = 4 - 4\sqrt{3}i$ in polar form.

Let's start by finding $|z|$. This is always easier. We compute: $|z| = \sqrt{16 + (16)(3)} = \sqrt{64} = 8$.

Now, we can see that z forms a right angle triangle with the positive real axis which has hypotenus 8, and side lengths 4 and $4\sqrt{3}$. If we view this as:



Then we have that $\Psi = -\theta$ and that:

$$\cos\Psi = \frac{4}{8} = \frac{1}{2}$$
$$\sin(\Psi) = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$$

We know from our special triangles that this gives $\Psi = \frac{\pi}{3}$. Therefore, $\theta = -\frac{\pi}{3}$.

So, in polar form, we have $4 - 4\sqrt{3}i = 8\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right)$.

There is another bit of notation, which you may have come across, used to write polar form.

Definition 1.4.2: Euler's Formula

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

So, instead of writing $z = r(\cos(\theta) + i\sin(\theta))$, we will from now on write $z = re^{i\theta}$. We will discuss why we use this notation (why e?), later on in the course. There is a reason.

Example 1.4.3

Find the polar form for $z = \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, and write it as $z = re^{i\theta}$.

We know that $r = |z|$. So we compute:

$$|z| = \sqrt{\left(\frac{-1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1} = 1$$

As for the angle, we need to be careful here. If we were to take the angle $\theta = \arctan\left(\frac{y}{x}\right) = \arctan(-1)$, we would end up with an angle in the fourth quadrant. But our complex number is in the second quadrant. So how do we handle this?

Well, note that our angle θ and $\arctan(-1)$ are on directly opposite sides of the unit circle, and so $\theta = \arctan(-1) + \pi = \frac{3\pi}{4}$.

Therefore, $z = 1e^{i\frac{3\pi}{4}}$.

Example 1.4.4

True or false: the real part of $3e^{i\frac{\pi}{2}}$ is 3.

False. This is a mistake I have seen quite a lot. You need to be able to find the real and imaginary parts of a complex number written in polar form.

In this case, $3e^{i\frac{\pi}{2}} = 3\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 3(0+i) = 3i$. So the real part of this complex number is 0!

Polar coordinates are useful in a variety of ways, which are majorly different from the ways in which rectangular form is useful. One way is that working in polar form makes doing multiplication very easy.

Theorem 1.4.1: Multiplication in Polar Form

Let $z = re^{i\theta}$ and $w = Re^{i\Psi}$. Then:

$$zw = rRe^{i(\theta+\Psi)}$$

Proof. We go back to rectangular coordinates. $z = r(\cos(\theta) + i\sin(\theta))$ and $w = R(\cos(\Psi) + i\sin(\Psi))$. Then:

$$\begin{aligned}
zw =& rR(\cos(\theta) + i\sin(\theta))(\cos(\Psi) + i\sin(\Psi)) \\
=& rR\left([\cos(\theta)\cos(\Psi) - \sin(\theta)\sin(\Psi)]\right. \\
&\left. + i\,[\sin(\theta)\cos(\Psi) + \cos(\theta)\sin(\Psi)]\right) \\
=& rR(\cos(\theta+\Psi) + i\sin(\theta+\Psi) \qquad \text{(trig identities)} \\
=& rRe^{i(\theta+\Psi)}
\end{aligned}$$

□

A similar argument can also be used to prove:

Theorem 1.4.2: Division in Polar Form

Let $z = re^{i\theta}$ and $w = Re^{i\Psi} \neq 0$. Then:

$$\frac{z}{w} = \frac{r}{R}e^{i(\theta-\Psi)}$$

Proof. The proof is similar to that of theorem 1.4.1, except using the angle difference formulas for sin and cos. Consider working through the proof, mimicing the proof of theorem 1.4.1. □

Note. The key ingredients in these proofs are trig identities. Specifically, the angle sum (and angle difference) formulas. Trig is very important to working with complex numers. You will need to know your trig identities.

Since multiplying in polar form is very quick, taking powers should also be really quick. Intuitively, the argument above tells us that if θ is an angle for z, then $n\theta$ is an angle for z^n. After all, if we add n copies of θ, we get $n\theta$. This intuitive idea is actually a named theorem!

Theorem 1.4.3: De Moivre's Theorem

Let $n \in \mathbb{N}$. Then $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$.

Proof. We proceed by induction. The claim is clearly true for $n = 1$.

Suppose $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$. This really just says: $(e^{i\theta})^n = e^{i(n\theta)}$.

Now, consider $(\cos(\theta) + i\sin(\theta))^{n+1}$. We have:

$$\begin{aligned}
(\cos(\theta) + i\sin(\theta))^{n+1} &= (e^{i\theta})^{n+1} \\
&= (e^{i\theta})^n e^{i\theta} \\
&= e^{i(n\theta)} e^{i\theta} && \text{(by the induction hypothesis)} \\
&= e^{i(n+1)\theta} && \text{(by theorem 1.4.1)} \\
&= \cos((n+1)\theta) + i\sin((n+1)\theta)
\end{aligned}$$

□

We end our discussion of complex algebra with one last definition. We will need to talk about the polar form, and specifically the angle, of a complex number very often.

Definition 1.4.3: The Argument

Let $z = re^{i\theta}$ be non-zero. The angle θ is called an argument for z.

We do not define an argument for $z = 0$.

The argument of a complex number is not unique. For example, $e^{i0} = 1$, and $e^{i2\pi} = \cos(2\pi) + i\sin(2\pi) = 1$. This means that 0 and 2π are both arguments for 1!

Example 1.4.5

Is $\frac{7\pi}{3}$ an argument for $1 + \sqrt{3}i$?

There are two approaches to this. One way would be to find an argument for $1 + \sqrt{3}i$. We recognize this as appearing on a 30-60-90 special triangle of hypotenuse 2, in the first quadrant. In particular, $\theta = \frac{\pi}{3}$ is an argument for $1 + \sqrt{3}i$.

Then we quickly check that $\frac{\pi}{3}$ and $\frac{7\pi}{3}$ point in the same direction, since they differ by a multiple of 2π.

Another approach would be to see what $e^{i\frac{7\pi}{3}}$ is. We find that $e^{i\frac{7\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, and so $1 + \sqrt{3}i = 2e^{i\frac{7\pi}{3}}$. So $\frac{7\pi}{3}$ is an argument for $1 + \sqrt{3}$.

The non-uniqueness of the argument ends up giving the complex numbers a lot of rich theory. For example, every non-zero number will have n different n^{th} roots. Every complex number will have infinitely many logarithms. We'll see this when we talk about the concept of "branches".

Sometimes, we don't need all that freedom. Very often, it's enough to consider arguments within a specific range. One particular choice is $(-\pi, \pi)$.

Definition 1.4.4: The Principal Argument

Let $z \in \mathbb{C}$ such that z is not a negative real number. The principal arugment of z is the argument $\text{Arg}(z) \in (-\pi, \pi)$.

Note. This is a different covention that some other sources you may encounter. I am specifically excluding negative reals from having a principal argument. I am not doing this arbitrarily however: this will allow us to avoid some ugly continuity issues later on when we define the principal logarithm, or other principal branches of multivalued functions.

1.5 n^{th} roots

We now turn our attention to square and higher roots. What is an nth root, and how do we find them?

Definition 1.5.1: n^{th} Roots

Let $n \in \mathbb{N}$. Then we say that z is an nth root of w if $z^n = w$.

In the real numbers, solving the equation $x^n = c$ is fairly straightforward. If n is even, then it has no solution for $c < 0$. And for $c \geq 0$, the solutions are $x = \pm\sqrt[n]{c}$. For n odd, there is always a unique solution: $x = \sqrt[n]{c}$.

We have already seen that this is no longer true for complex numbers. In particular, the equation $z^2 = -1$ has a solution: i (and $-i$ as well). This is true in a much more broad sense. The equation $z^n = c$ always has a solution, and De Moivre's theorem tells us exactly how to find such solutions.

Let us begin with an example, to see the general tactic in action.

Example 1.5.1

Find all square roots of $z^2 = 1 + i$.

There are a couple of ways to approach this. One way is to write $z = x + iy$, and then expand $z^2 = 1 + i$ to get the equations:

$$x^2 - y^2 = 1$$

$$2xy = 1$$

Now, this is solvable. For example, we can write $y = \frac{1}{2x}$, and substitute this into the first equation, giving $x^2 - \frac{1}{4x^2} = 1$. Rearranging to give $4x^4 - 4x^2 - 1 = 0$. We can then use the quadratic formula to find x.

However, this approach has some major drawbacks. This isn't an easy calculation, to start. But worse, it doesn't generalize. For example, if we wanted to solve $z^3 = 1 + i$, we would need to solve the system $x^3 - 3xy^2 = 1$ and $3x^2y - y^3 = 1$, which is quite a bit more difficult. This approach doesn't

work for higher powers.

Instead, let's see what happens if we work in polar form. Let $z = re^{i\theta}$. Then we have:

$$r^2 e^{2i\theta} = 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$$

By comparing moduli on both sides, we find $r^2 = \sqrt{2}$, so $r = \sqrt[4]{2}$.

Also, by comparing arguments, we see that 2θ is an argument for $1+i$. I.e., $2\theta = \frac{\pi}{4} + 2k\pi$ for some $k \in \mathbb{Z}$.

As such, $z = \sqrt[4]{2}e^{i\frac{\pi}{8}+k\pi}$ for some $k \in \mathbb{Z}$. Recalling that $e^{i\theta}$ is 2π periodic, we see that we only get two distinct solutions: $\pm\sqrt[4]{r}e^{i\frac{\pi}{8}}$.

Does this work generally? Is there some algorithm or formula that gives us n^{th} roots?

Theorem 1.5.1: Existence of nth Roots

Let $w = re^{i\theta} \in \mathbb{C}$. If $w = 0$, then $z^n = w$ has a unique solution, $z = 0$.

If $w \neq 0$, then the solutions to $z^n = w$ are:

$$z_j = \sqrt[n]{r}e^{\frac{i(\theta+2j\pi)}{n}}$$

where $j \in \mathbb{Z}$. Furthermore, it is enough to assume $j \in \{0, 1, 2, \ldots, n-1\}$.

Proof. When $w = 0$, the claim is clear.

For $w \neq 0$, what do we need to show? We need:

- The z_j are solutions to $z^n = w$. I.e., $z_j^n = w$.
- The z_j are the only solutions to $z^n = w$. I.e., if $z^n = w$, then $z = z_j$ for some j.

In other words, we need to prove that $z^n = w$ if and only if $z = z_j$ for some j. Let $z = se^{i\Psi}$ be in polar form. Then $z^n = w \iff s^n e^{in\Psi} = re^{i\theta}$. This occurs if and only if $s^n = r$ and $n\Psi = \theta + 2k\pi$ for some $k \in \mathbb{Z}$, by considering the moduli and arguments of each side.

As such, $z^n = w$ if and only if $s = \sqrt[n]{r}$ and $\Psi = \frac{\theta+2k\pi}{n}$ for some $k \in \mathbb{Z}$. I.e., $z^n = w$ if and only if $z = z_k$ for some $k \in \mathbb{Z}$.

Lastly, we need to justify why we only need to consider $j \in \{0, 1, \ldots, n-1\}$. Let $k \in \mathbb{Z}$. Then by the division algorithm, we can write $k = qn + j$ for some $0 \leq j \leq n-1$. We find that:

$$z_k = \sqrt[n]{r}e^{i\frac{\theta+2k\pi}{n}} = \sqrt[n]{r}e^{i\frac{\theta+2(qn+j)\pi}{n}} = \sqrt[n]{r}e^{i\frac{\theta+2j\pi}{n}+2q\pi i} = \sqrt[n]{r}e^{i\frac{\theta+2j\pi}{n}} = z_j$$

As such, each z_k is actually equal to some z_j, where $0 \leq j \leq n-1$.

Also, note that each of the z_j for $j \in \{0, 1, \ldots, n-1\}$ are distinct. Since $\frac{\theta+2j\pi}{n} \in \left[\frac{\theta}{n}, \frac{\theta}{n} + 2\pi\right)$, we see that these angles all point in different directions.

$\square$

Example 1.5.2

Let $w = i$. Find all solutions to $z^2 = w$ and $z^4 = w$.

To begin, we need to write w in polar form. In this case, it is simple: $w = e^{i\frac{\pi}{2}}$. The theorem gives us a formula for these roots.

To solve $z^2 = w$, we consider $z_0 = \sqrt{1}e^{i\frac{\pi/2}{2}} = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$.

The other root is much easier to find. Notice that the other root is $z_1 = e^{i\frac{\pi/2+2\pi}{2}} = e^{i\frac{\pi/2}{2}}e^{\frac{2\pi}{2}} = -z_0$.

In solving $z^4 = w$, we can take a similar approach. Indeed, we have that $z_j = z_0 e^{i\frac{2j\pi}{4}}$. This gives us a list:

- $z_0 = e^{i\frac{\pi}{8}}$
- $z_1 = z_0 e^{i\frac{2\pi}{4}} = iz_0$
- $z_2 = z_0 e^{i\frac{2*2\pi}{4}} = -z_0$
- $z_3 = -iz_0$

As an interesting aside, it turns out that $e^{i\frac{\pi}{8}} = \frac{\sqrt{2+\sqrt{2}}}{2} + i\frac{\sqrt{2-\sqrt{2}}}{2}$. One possible way to show this would be to try to solve the equation $z^2 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ by setting $z = a + bi$.

Notice, in our example, we factored z_j as $\sqrt[n]{r}e^{i\frac{\theta}{n}}e^{i\frac{2j\pi}{n}}$. These numbers $e^{i\frac{2j\pi}{n}}$ are the nth roots of unity.

Definition 1.5.2: Roots of Unity

The nth roots of unity are the solutions to $z^n = 1$. They are precisely the complex numbers $\omega_j = e^{i\frac{2j\pi}{n}}$.

1.6 Geometry of roots

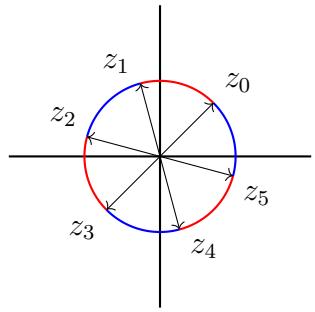
There are two nice geometric facts about the nth roots of w that we can glean from the results of the previous section.

- Notice that $|z_j| = \sqrt[n]{r}$ for each j. This means that each of the nth roots of w all lay on the circle of radius $\sqrt[n]{r}$ centered at 0.

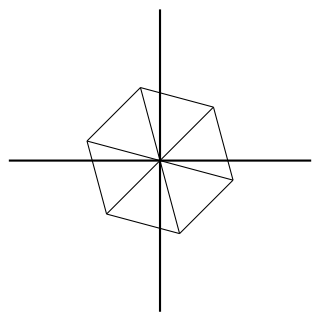
  This circle has the equation $|z| = \sqrt[n]{r}$. More generally, a circle of radius r centered at z_0 has the equation $|z - z_0| = r$.

- Notice that the angle between z_j and z_{j+1} is exactly $\frac{2\pi}{n}$.

  So, for example, the 6th roots of w form a picture like:



Where the arc between z_j and z_{j+1} has angle $\frac{\pi}{3}$. Notice that this is a regular hexagon!



And this is true in general. If $w \neq 0$ and $n \geq 3$, then the roots of $z^n = w$ form a regular n-gon.

I made a small note in the preceeding discussion which I would like to expand upon. Namely, how to describe complex circles.

Theorem 1.6.1

The set of points $\{z \in \mathbb{C} | |z - z_0| = r\}$ is a circle of radius r centered at z_0.

Proof. The circle of radius r centered at (x_0, y_0) in $\mathbb{R}^2$ is described by the equation $(x - x_0)^2 + (y - y_0)^2 = r^2$. We wish to show that $|z - z_0| = r$ is equivalent to this expression.

Suppose $z = x + iy$ and $z_0 = x_0 + iy_0$. Then:

$$r = |z - z_0| = |(x - x_0) + i(y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Squaring both sides gives the desired result. $\square$

1.7 Functions

Our major goal is to talk about complex calculus: both differentiation and integration. To do so, we need some notion of a function.

Definition 1.7.1: Function

Let U, W be subsets of $\mathbb{C}$. A function $f : U \to W$ is a rule that assigns to each element $z \in U$ exactly one element $f(z) \in W$. U is called the domain of f, and W is called the codomain of f.

The range of f is the set $\{f(z) | z \in U\}$. The range does not need to be all of W.

So, in essence, functions of a complex variable are defined exactly the same way any other function is. The difference here is that the domain and codomain both lie in planes, so we won't be able to draw graphs to visualize functions. After all, how do you visualize a 4-dimensional picture? Actually, we also aren't going to be able to use contour maps to understand these functions either

Example 1.7.1

Find the domain and range of the function $f(x + iy) = \frac{ix+y}{x-iy}$.

When finding the domain of a function given by a formula, take the largest set on which that formula makes sense. So, in this example, the formula gives an output whenever $x - iy \neq 0$. I.e., when $z \neq 0$. So, the domain of this function is $\mathbb{C} \setminus \{0\}$.

For the range, w is in the range if there exists z with $f(z) = w$. So, we have to solve the equation $\frac{ix+y}{x-iy} = w$.

There are two ways to do this, depending on whether you notice something.

Hard way: Set $w = a + ib$. Then we want:

$$ix + y = (x - iy)(a + ib) = xa + yb + i(xb - ay)$$

So, $f(z) = w$ for some z if there exist x, y making this equation true.

If we look at the real and imaginary parts of this equation, we find that:

$$x = xb - ay$$

$$y = xa + yb$$

Alternatively, we can rewrite this as:

$$(b - 1)x - ay = 0$$

$$ax + (b - 1)y = 0$$

This is a system of linear equations in two variables, so we can solve this:

$$\left[\begin{array}{cc|c} b-1 & -a & 0 \\ a & b-1 & 0 \end{array}\right]$$

Note, this matrix has determinant $(b - 1)^2 + a^2$. If the determinant is non-zero, we have a unique solution $x = y = 0$. This would give $z = 0$, which is not in our domain.

If the determinant is zero, then there is a non-trivial solution. So if $(b - 1)^2 + a^2 = 0$, $w = a + ib$ is in the range. This occurs exactly when $a = 0$ and $b = 1$. I.e., when $w = i$.

Easy way: Notice that $ix + y = i(x - iy)$. So, if $z \neq 0$, then $f(z) = i$. The range of f is $\{i\}$.

As this example illustrates, generally it is difficult to find the range of a complex function. But this isn't terribly different than working over the reals. Except for the very simple functions we see in first year calculus, it is also generally hard to find the range of a real function as well. Especially functions whose domain is in $\mathbb{R}^2$. The methods we used in this example do not generalize; finding ranges is always ad hoc.

Next, let's take a little tour of a few of the basic functions we're going to encounter. First, polynomials:

Definition 1.7.2: Polynomials

A polynomial p on $\mathbb{C}$ is a function of the form:

$$p(z) = a_n z^n + \cdots + a_1 z + a_0 = \sum_{k=0}^{n} a_k z^k$$

The degree of p is the largest n such that $a_n \neq 0$.

We're going to be spending a decent amount of time talking about polynomials in this course. Keep in mind that these don't behave the same way as real polynomials, and that we can have complex coefficients. $p(z) = z^2 + iz - (1 - i)$ is a polynomial.

Let's also talk about root functions. Over the reals, it's fairly easy to define root functions: either x has 0 nth roots, 1 nth root, or 2 nth roots. If it has no nth roots, then $f(x)$ isn't defined. If it has 1 nth root, then that's $f(x)$. And if it has 2 roots, then one is positive and we choose that to be $f(x)$.

This isn't possible over $\mathbb{C}$. We have n nth roots, and we don't have any notion of positivity. (Notice, we've never talked about $z < w$ where z, w are complex numbers. That's because it's not possible to define in a useful way.)

Note. There is no notion of a positive complex number, and it does not mean anything to say that $z < w$ for complex numbers.

In this text, if we write $a < b$, it must be understood that a, b are **real numbers.**

So do we have an nth root function on $\mathbb{C}$? Let's start by taking a look at a bad way to define such a function:

Example 1.7.2

Consider the following (clearly false) proof:

Claim. $1 = -1$.

Proof. Let $z = re^{i\theta}$. Consider the function $f(z)$ defined by $f(z) = \sqrt{r}e^{i\frac{\theta}{2}}$. Note that $f(z)^2 = z$, and so $f(z)$ is a square root function.

As we have already seen, we can write $1 = e^{i0} = e^{i(2\pi)}$. Applying our function gives:

$$f(1) = f(e^{i0}) = \sqrt{1}e^{i\frac{0}{2}} = 1e^{i0} = 1$$
$$f(1) = f(e^{i(2\pi)}) = \sqrt{1}e^{i\frac{2\pi}{2}} = e^{i\pi} = -1$$

Therefore, $1 = f(1) = -1$, as desired. □

Exercise

What's wrong with this argument?

Well, if you take me at my word that $f(z)$ is a function, nothing. So that's the problem, $f(z)$ isn't a function. Not every formula defines a function, so we need to be careful.

In fact, our argument here really just shows that this formula doesn't define a function.

So, if we want to define an nth root function, we need to be a lot more careful. The nth root is our first example of a common theme with complex formulae: it's a multivalued function.

Definition 1.7.3: Multi-valued Function

A **multi-valued function** f on $\mathbb{C}$ is a rule that assigns to $z \in \mathbb{C}$ a set of (possibly more than one) outputs.

The output set of f is still written as $f(z)$.

For example, the formula $f(re^{i\theta}) = \sqrt{r}e^{i\frac{\theta}{2}}$ is a rule that assigns to $z \neq 0$ its two square roots. It therefore defines a multi-valued function. We can similarly look at all n^{th} roots as giving multi-valued function. The notation for this is:

Definition 1.7.4: $z^{\frac{1}{n}}$

Let $z = re^{i\theta}$. The formula $f(z) = \sqrt[n]{r}e^{i\frac{\theta}{n}}$ defines a multi-valued function on $\mathbb{C}$. We denote this function by $z^{\frac{1}{n}}$.

We need to be careful now. Our goal is to do calculus. That's going to require us to have functions to work with. So how can we go from having a multi-valued function to an actual function?

Definition 1.7.5: Branch of a Multi-valued Function

Let f be a multivalued function with domain U. A **branch** of f is a function $g : U \to \mathbb{C}$ such that $g(z) \in f(z)$. (Remember that $f(z)$ is a set, so this makes sense.)

So, for each input, we pick one output (out of the possible outputs given by the multi-valued function) to be the output of the branch. Now, without care, you can choose some truely bizarre branches. For example, for the square root, we could say that if $z = x + iy$, we choose the square root $a + bi$ with $a \geq 0$ if x is rational, and if x is irrational we choose the square root with $a < 0$.

This is a contrived example, but that's the point. You can cook up some truly weird and unpleasant

branches if you set your mind to it. Is there some way to choose our branches nicely? For some nice functions yes, and it comes down to the argument of z, actually.

Example 1.7.3

Let $z \in \mathbb{C}$. For $z \neq 0$, we define $\arg(z) = \{\theta \in \mathbb{R} | z = |z|e^{i\theta}\}$ (i.e., $\arg(z)$ is the set of all arguments of z). Then this is a multi-valued function (in fact, infinitely-valued) function on $\mathbb{C}$.

One way to take branches of arg is to specify a range of angles. So, for example, we could get a branch of the argument $\arg_0(z)$ by choosing $\arg_0(z) = \theta$ where θ is the unique angle in $\arg(z) \cap [0, 2\pi)$. This rule assigns to each $z \neq 0$ a single argument.

Example 1.7.4

Let $\theta \in \mathbb{R}$. Define $\arg_0(z)$ to be the branch of $\arg(z)$ with $\arg_0(z) \in [\theta, \theta + 2\pi)$.

Then for $z = re^{i\theta}$, we can define $z^{\frac{1}{2}} = \sqrt{r}e^{i\frac{\arg_0(z)}{2}}$. This gives a branch of the square root.

Notice that $(z^{\frac{1}{2}})^2 = \sqrt{r}^2(e^{i\frac{\arg_0(z)}{2}})^2 = re^{i\frac{2\arg_0(z)}{2}} = re^{i\arg_0(z)} = z$. So this is actually a square root.

It also only gives one output to each input. Each $z \in \mathbb{C}$, except $z = 0$, has a unique argument $\arg_0(z)$ between θ and $\theta + 2\pi$. As such, the formula doesn't depend on the angle we choose for z. Indeed, there is no choice!

This may seem a bit arcane. Frankly, this is one of the more difficult concepts in this course. We're going to come back to it once we introduce the complex logarithm (which will also be a multi-valued function). I wanted to introduce the concept before we run into that.

Notation: Very often, we will write equations involving multi-valued functions. For example, we can write the quadratic formula as:

$$z = \frac{-b + (b^2 - 4ac)^{\frac{1}{2}}}{2a}$$

It is important to understand that we are saying that z takes on two different values, one for each different value of the square root.

1.7.1 The Exponential Function

In definition 1.4.2, we defined $e^{i\theta}$ for any $\theta \in \mathbb{C}$. We can use this to give a definition of e^z.

Definition 1.7.6: The Exponential Function

Let $z = x + iy$. Then we define the exponential e^z as:

$$e^z = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

Unlike roots, this is a function. Indeed, for each $z \in \mathbb{C}$, there is a unique choice of $x, y \in \mathbb{R}$ such that

$z = x + iy$, and so we don't get multiple values coming from this formula.

Is this a good definition? What might we expect to be true of a complex exponential function. We would like:

- $e^z e^w = e^{z+w}$
- $e^z \neq 0$
- For $z = r \in R$, we would like e^z (the complex exponential) to be equal to e^r (the real exponential).
- If $z = iy$, we should have that $e^z = \cos(y) + i\sin(y)$, so that this formula also agrees with Euler's formula.
- It should be a differentiable function, once we've defined what that means in $\mathbb{C}$.

And these turn out to all be true! The third and fourth are a quick exercise in using the definition. We'll prove the first very quickly:

Proof. Let $z = x + iy$ and $w = a + ib$. Then:

$$\begin{aligned} e^z e^w &= e^x e^{iy} e^a e^{ib} \\ &= e^x e^a e^{i(y+b)} && \text{(by theorem 1.4.1)} \\ &= e^{x+a} e^{i(y+b)} \end{aligned}$$

This is exactly e^{z+w} since $z + w = (x + a) + i(y + b)$. □

Example 1.7.5

Let's compute a few exponentials. Find e^{1+i} and e^{1-i}.

Well, we compute:

$$e^{1+i} = e^1 e^i = e^1(\cos(1) + i\sin(1)$$
$$e^{1-i} = e^1 e^{-i} = e^1(\cos(-1) + i\sin(-1) = e^1(\cos(1) - i\sin(1))$$

Did you notice anything about these two numbers? Give a conjecture for the relationship between e^z and $e^{\overline{z}}$.

Example 1.7.6

A function f is called **injective** if $f(z) = f(w)$ implies $z = w$.

Exercise

True or false: $f(z) = e^z$ is an injective function.

False. We have already seen that $e^0 = e^{i2\pi}$. In fact, for any $w \neq 0$, $e^z = w$ has infinitely many solutions!

Example 1.7.7

Let $w = 1 + 4i$. Solve the equation $e^z = w$.

Let $z = x + iy$. Then $e^x e^{iy} = 1 + 4i$. This tells us that:

$$e^x = |w| = \sqrt{17}$$

So we conclude that $x = \ln(\sqrt{17})$.

Further, the expression $w = e^x e^{iy}$ tells us that y is an argument for w! So, we need to find the arguments for w. We see that $w = \sqrt{17}e^{i\arctan(4)}$.

Therefore, $z = \ln(\sqrt{17}) + i\arctan(4)$.

Exercise

There is an error in this argument. What is it?

We know that y is AN argument for w. Not that y is this particular argument for w. Instead, we can only conclude that $y = \arctan(4) + 2k\pi$ for some $k \in \mathbb{Z}$, and therefore $z = \ln(\sqrt{17}) + i(\arctan(4) + 2k\pi)$.

Note. $\ln(\sqrt{17}) + i\arctan(4)$ and $\ln(\sqrt{17}) + i(\arctan(4) + 2\pi)$ are different complex numbers! While $\arctan(4)$ and $\arctan(4) + 2\pi$ point in the same direction as angles, y is not an angle. y is the vertical component of the complex number z.

This distinction is important. y is not an angle, and so we can't ignore this $2k\pi$.

Example 1.7.8

Solve the equation $e^{iz} - e^{-iz} = 2i$.

It is possible to solve this equation by setting $z = x + iy$, expanding into rectangular form, and then solving the resulting equations. However, this turns out to be fairly difficult.

Instead, let $e^{iz} = w$. Then we have:

$$w - \frac{1}{w} = 2i$$

After some quick algebra, we can rearrange this to become:

$$w^2 - 2iw - 1 = 0$$

Now, from your homework, we know that we can solve this using the quadratic formula:

$$w = \frac{2i \pm (-4 + 4)^{\frac{1}{2}}}{2} = i$$

So, the solutions z to the original equation satisfy $e^{iz} = i$. Let $z = x + iy$. Then $e^{-y+ix} = i$. This gives $e^{-y} = 1$, so $y = 0$.

Also, we know that x is an argument for i, so $x = \frac{\pi}{2} + 2k\pi$ for $k \in \mathbb{Z}$. Therefore, $e^{iz} - e^{-iz} = 2i$ if and only if $z = \frac{\pi}{2} + 2k\pi$ for some $k \in \mathbb{Z}$.

Example 1.7.9

True or false: For any $z \in \mathbb{C}$, $e^z > 0$.

False. Remember, it is nonsense to say that $a > b$ if a, b are complex numbers. e^z can be any complex number (except for 0), so this is a nonsense statement.

More concretely, $e^{i\pi} = -1 \not> 0$.

1.7.2 Complex Trigonometric Functions

Looking at Euler's formula, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, it seems like there's a connection between trigonometric functions and the complex exponential.

If we play around with this fact, we can see:

$$\begin{aligned} e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\ e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta) \end{aligned}$$

Adding these expressions together, we see that:

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Since this is our only connection between trigonometric functions and complex numbers, it seems reasonable to use this to define complex versions of $\cos(z)$ and $\sin(z)$.

Definition 1.7.7: Trigonometric Functions

The complex differentiable functions $\cos(z)$ and $\sin(z)$ are defined by:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

The other trigonometric functions $\tan(z)$, $\sec(z)$, $\csc(z)$, and $\cot(z)$ are defined exactly how you would expect.

Just like for our complex exponential, notice that if $z = x \in \mathbb{R}$, then $\cos(z)$ is exactly the real cosine function $\cos(x)$, and similarly for $\sin(z)$. So this isn't totally unreasonable.

Many of the usual properties of the real trigonometric functions are still satisfied by these complex functions as well. For example:

Example 1.7.10

We know that the real function $\cos(x)$ is 2π periodic. I.e., $\cos(x) = \cos(x + 2\pi)$. This is still true over $\mathbb{C}$.

To see this, note that:

$$\cos(z + 2\pi) = \frac{e^{i(z+2\pi)} + e^{-i(z+2\pi)}}{2} = \frac{e^{iz}e^{i2\pi} + e^{-iz}e^{-i2\pi}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \cos(z)$$

However, these complex functions can have some wildly different behavior as well.

Example 1.7.11

True or false: $-1 \leq |\sin(z)| \leq 1$ for any $z \in \mathbb{C}$.

As we saw in class, when looking at $\sin(iy)$, we see that for y very large, $|\sin(iy)|$ is very large. In fact, $|\sin(iy)| \approx \frac{e^{|y|}}{2}$.

In fact, the range of $\sin(z)$ is actually $\mathbb{C}$!

Example 1.7.12

Find all solutions to $\sin(z) = 1$.

We are looking for all z for which $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i} = 1$. Notice that this is equivalent to solving $e^{iz} - e^{-iz} = 1$. As we saw in example 1.7.8, the solutions to this are precisely $z = \frac{\pi}{2} + 2k\pi$ for $k \in \mathbb{Z}$. So the only solutions to $\sin(z) = 1$ are real solutions.

1.7.3 The Complex Logarithm

We have a notion of complex exponentiation. Do we have a corresponding notion of complex logarithms?

To begin, what is a logarithm? What does it mean to say that w is a logarithm for z?

Definition 1.7.8: Logarithms

Let $z \in \mathbb{C}$. We say that w is a logarithm for z if $e^w = z$.

We've already seen an example of finding a logarithm. Last class, we showed that the solutions to $e^z = 1+4i$ are of them form $z = \ln(\sqrt{17}) + i(\arctan(4) + 2k\pi)$ for $k \in \mathbb{Z}$. So, $1 + 4i$ has many logarithms. These are precisely the z listed above.

Is there a general formula for finding the logarithms of $z \in \mathbb{C}$, or do we need to do it by hand each time we wish to solve $e^z = w$?

Theorem 1.7.1: Calculating Logarithms

Let $z = re^{i\theta}$ with $z \neq 0$. Then the logarithms of z are the complex numbers $\ln(r) + i(\theta + 2k\pi)$, where $k \in \mathbb{Z}$. Put another way, $\log(z) = \ln|z| + i\arg(z)$, remembering that we mean this as multi-valued functions.

If $z = 0$, then z has no logarithms.

Proof. First, we handle the situation where $z \neq 0$. Suppose $w = a + bi$ and $e^w = z$. Then:

$$e^a e^{ib} = re^{i\theta}$$

Taking the modulus of both sides, we see that $e^a = r$, and so $a = \ln(r)$, which is defined since $r \neq 0$.

Now, we can see that $re^{ib} = z$, and so b is an argument for z. As we have shown before, this means that $b = \theta + 2k\pi$ for some $k \in Z$.

As for $z = 0$, notice that $|e^w| = e^a \neq 0$. However, $|z| = 0$. So, we cannot have $e^w = z$.

□

The complex logarithm is the most important example of a multi-valued function. In fact, all of the examples we are going to see in this course (including the ones we already have seen!) will depend on the complex logarithm.

Notation. We are often very lazy with our notation for logarithms. If $e^w = z$, we very often write that $w = \log(z)$.

But, as we've seen, it is possible to have $w_1 \neq w_2$ with $e^{w_1} = e^{w_2} = z$. So is $w_1 = \log(z)$ or is $w_2 = \log(z)$?

The answer is that $\log(z)$ isn't really one number. The complex logarithm is a multi-valued function, and so w_1 and w_2 are two different values of the same multi-valued function. So when we say that $w = \log(z)$, we really mean that w is **one of** the logarithms of z.

Definition 1.7.9: $\log(z)$

The complex logarithm $\log(z)$ is the multi-valued function:

$$\log(z) = \{w \in \mathbb{C} | e^w = z\}$$

For any $z \neq 0$, $\log(z)$ is infinitely-valued.

Example 1.7.13

Suppose we know that $\log(z) = 1 + 3i$. Is it possible that $\log(z) = 1 + 7i$?

No. We would need $e^{1+3i} = e^{1+7i}$. But these have different angular components. The angles 3 and 7 do not point in the same direction!

Definition 1.7.10: Complex Exponentiation

Let $a, z \in \mathbb{C}$. Then $z^a = e^{a\log(z)}$.

How do we interpret this? After all, we just discussed that $\log(z)$ isn't one number. This formula should be interpretted as saying that z^a is a multi-valued function, and that its values are e^{aw} where w is a logarithm for z.

Example 1.7.14

This definition has some surprising consequences. For example, every value of i^i is real!

Why is that? Well, $i^i = e^{i\log(i)}$. However, since $|i| = 1$, we see that the logarithms of i are:

$$\log(i) = \ln(1) + i\left(\frac{\pi}{2} + 2k\pi\right) = i\left(\frac{\pi}{2} + 2k\pi\right)$$

As such, $i^i = e^{i^2\left(\frac{\pi}{2}+2k\pi\right)} = e^{-\frac{\pi}{2}+2k\pi}$, which is a real number!

Example 1.7.15

Consider the following claim:

Fake Theorem 1.7.1

Every complex number is real.

You should quickly convince yourself that this is false. For example, why is i not real? (What property defines i?)

So, if this is a false claim, any proof of this claim must have an error. Find the error (or errors, if there are more than one) in the following proof.

Proof. Let $z \in \mathbb{C}$, and write z in polar form as $z = re^{i\theta}$. Then we find that:

$$z = re^{i\theta} = re^{i\left(\frac{2\pi\theta}{2\pi}\right)} = r(e^{2\pi i})^{\frac{\theta}{2\pi}}$$

But $e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$. So:

$$z = r(1^{\frac{\theta}{2\pi}}) = r \in \mathbb{R}$$

Since $z \in \mathbb{R}$ for any complex number z, every complex number is real. □

Be careful. Any errors are subtle. If you think you have an easy answer, chances are your answer is not correct.

Solution: As we discussed in class, the error occurs in two places. When we write:

$$e^{i\frac{2\pi\theta}{2\pi}} = (e^{i2\pi})^{\frac{\theta}{2\pi}}$$

we are choosing a branch $f(z)$ of the multivalued function $z^{\frac{\theta}{2\pi}}$ so that $f(1) = e^{i\theta}$.

On the other hand, when we say that $1^{\frac{\theta}{2\pi}} = 1$, we are choosing a branch $g(z)$ with $g(1) = 1$. We're working with two different branches as if they are the same!

Now that we have talked about how they are multivalued, and how that requires some care, let's talk about their branches. Corresponding to the principal Argument $\text{Arg}(z)$, there is a principal branch of these multivalued functions as well:

Definition 1.7.11: Principal Logarithm

Let $z \in \mathbb{C} \setminus (-\infty, 0]$. The principal logarithm of z is:

$$\text{Log}(z) = \ln|z| + i\text{Arg}(z)$$

Definition 1.7.12: Principal Branch of z^a

The principal branch of z^a is given by $e^{a\text{Log}(z)}$.

Example 1.7.16

Find the principal value of i^{1-i}.

The principal value (which comes from the principal branch) is:

$$e^{(1-i)\text{Log}(i)} = e^{(1-i)i\frac{\pi}{2}} = e^{\frac{\pi}{2}+i\frac{\pi}{2}} = e^{\frac{\pi}{2}}i$$

Example 1.7.17

Let $n \in \mathbb{Z}$. Is z^n a single valued function?

Well, $z^n = e^{n\log(z)}$. Let $z = re^{i\theta}$. Then:

$$z^n = e^{n\log(z)} = e^{n(\ln|z|+i(\theta+2k\pi))}$$

where $k \in \mathbb{Z}$. However, notice that $nk \in \mathbb{Z}$, so $e^{i(2nk)\pi} = 1$. Therefore:

$$z^n = e^{n(\ln|z|+i\theta)} = e^{\ln(|z|^n)+in\theta} = |z|^n e^{in\theta}$$

So this is a single valued function. Regardless of our choice of argument, we get the same result.

Example 1.7.18

Consider the formula $f(z) = a^z$. Is this a function?

Let $z = x+iy$. We have $a^z = e^{z\log(a)} = e^{z(\ln|a|+i\arg(a))} = e^{(x\ln|a|-y\arg(a))+i(x\arg(a)+y\ln|a|)}$. This formula outputs a single value if and only if $y\arg(a)$ doesn't depend on the choice of argument, so $y = 0$. Also, we need that $e^{ix\arg(a)}$ doesn't depend on the choice of argument, so $x \in \mathbb{Z}$.

So, this tells us that a^z is a multi-valued function as well! Does that mean that e^z is multi-valued? The answer to that is no. Our definition of e^z doesn't depend on the argument of e. Technically, our definition of e^z is the principal branch of the function:

$$f(z) = e^{\mathrm{Re}(z\log(e))}(\cos(\mathrm{Im}(z\log(e))) + i\sin(\mathrm{Im}(z\log(e))))$$

However, to avoid unnecessary notational baggage (after all, this function is a bit of a mouthful) and to avoid unnecessary abstraction, e^z will always be understood to be an exception to a^z being a multi-valued function.

Example 1.7.19

Find the range of $\sin(z)$.

Let $w \in \mathbb{C}$. We would like to see if there exists some $z \in \mathbb{C}$ with $\sin(z) = w$. Supposing there is, we have:

$$\frac{e^{iz} - e^{-iz}}{2i} = w$$

Rearranging gives $e^{iz} - 2iw - e^{-iz} = 0$. Let $u = e^{iz}$. Since $u \neq 0$, we see that:

$$u - 2iw - \frac{1}{u} = 0 \iff u^2 - 2iwu - 1 = 0$$

And the quadratic formula tells us that:

$$u = iw + (-w^2 + 1)^{\frac{1}{2}}$$

And so $z = \frac{\log(u)}{i}$.

But wait, we're not done! We need to know that $\log(u)$ actually exists for any given w. We assumed $u \neq 0$, but are we guaranteed that it is? After all, it depends on w. How do we know there doesn't exist some w such that $u = 0$?

Suppose $u = 0$. Then $(1-w^2)^{\frac{1}{2}} = iw$. Squaring both sides gives $1-w^2 = -w^2$, which cannot occur. So $u \neq 0$, and therefore such a z exists for any w.

This example gives us an idea of how to define inverse trig functions as well:

Definition 1.7.13: arcsin

Let $z \in \mathbb{C}$. Then:

$$\arcsin(z) = -i\log(iz + (1-z^2)^{\frac{1}{2}})$$

Further, the principal arcsin is given by:

$$\mathrm{Arcsin}(z) = -i\mathrm{Log}(iz + (1-z^2)^{\frac{1}{2}})$$

where $(1-z^2)^{\frac{1}{2}}$ is the principal square root.

2 Limits and Differentiation

In this chapter, we will continue to develop the theory of complex functions in a way that mirrors your first introduction to calculus. Now that we know what our functions of interest are, let's talk about limits and differentiation.

2.1 Limits

Just like for differentiation over $\mathbb{R}$, our first building block is the limit. There is a complication here though: in $\mathbb{R}$, when we take a limit we're looking at two directions: $\lim_{x\to c} f(x)$ depends on what $f(x)$ does as x approaches c from the left hand and from the right hand.

In $\mathbb{C}$, we not longer have two directions. We have an infinite number! More than that though, we need to consider not just travelling along straight lines, but rather any curve toward our point.

To capture this, I'll present two definitions. The first is a fairly formal one, and we aren't going to work with it at all. It may appear in some proofs, but that's all. The second is the intuitive way to understand limits.

Definition 2.1.1: δ-ε Definition of a Limit

Let $f : U \to \mathbb{C}$, and $z_0 \in \mathbb{C}$. Further, assume there exists some $r > 0$ such that $\{z \in \mathbb{C} | 0 < |z - z_0| < r\} \subset U$. Then $\lim_{z\to z_0} f(z) = L$ if:

$$\forall \varepsilon > 0, \exists \delta \text{ such that } 0 < |z - z_0| < \delta \implies |f(z) - L| < \varepsilon$$

Intuitively, this says that for any $\varepsilon > 0$, we can find a circle around z_0 so that if z is inside this circle, then the distance from $f(z)$ to L is less than ε. I.e., $f(z)$ gets as close as we want to L, and stays that close.

There's another way to understand limits that is more in line with how we visualize limits. It involves looking at the real and imaginary parts of $f(z)$.

Definition 2.1.2: $\mathbb{R}^2$ Definition of a Limit

Let $f : U \to \mathbb{C}$ and $z_0 = x_0 + iy_0 \in U$. Further, assume there exists some $r > 0$ such that $\{z \in \mathbb{C} || z - z_0| < r\} \subset U$. Let $z = x + iy$, and write $f(x + iy) = u(x, y) + iv(x, y)$. That is, write $f(z)$ in terms of its real and imaginary parts, which we will view as functions on $\mathbb{R}^2$.

Then $\lim_{z\to z_0} f(z) = L$ if:

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = \mathrm{Re}(L)$$

$$\lim_{(x,y)\to(x_0,y_0)} v(x, y) = \mathrm{Im}(L)$$

So, in essence, complex limits can be viewed as just a pair of limits on $\mathbb{R}^2$. Remember, when looking at limits on $\mathbb{R}^2$, we need to consider arbitrary paths to z_0. So, for example, when taking a limit to 0, the limit must exist and be the same along $y = 0$, $x = 0$, $y = x$, $y = x^4$, etc.

Because of this, a lot of complex limits turn out to be unpleasant. However, unlike working over $\mathbb{R}^2$, a lot will turn out to be really nice. Many of the techniques for understanding limits that we saw in first year calculus work.

Example 2.1.1

Find $\lim_{z\to 0} \frac{z}{\overline{z}}$.

You could try doing this algebraically. However, this is much easier to understand by trying a few paths out.

If the limit exists, then its value must be give by approaching 0 along the line $y = 0$. We find:

$$\lim_{z\to 0} \frac{z}{\overline{z}} = \lim_{(x,0)\to(0,0)} \frac{x + 0i}{x - 0i} = 1$$

On the other hand, if the limit exists, it must also be given by approaching 0 along the line $x = 0$. We find:

$$\lim_{z\to 0} \frac{z}{\overline{z}} = \lim_{(0,y)\to(0,0)} \frac{0 + iy}{0 - iy} = -1$$

Since these limits disagree, we find that $\lim_{z\to 0} \frac{z}{\overline{z}}$ does not exist.

Example 2.1.2

Find $\lim_{z\to 0} e^z$.

As we have seen before, we know that $e^z = e^x \cos(y) + ie^x \sin(y)$. From our definition, we know that we need to find the limits:

$$\lim_{(x,y)\to(0,0)} e^x \cos(y)$$
$$\lim_{(x,y)\to(0,0)} e^x \sin(y)$$

For the first one, we can use the product law for limits to get:

$$\lim_{(x,y)\to(0,0)} e^x \cos(y) = \left(\lim_{(x,y)\to(0,0)} e^x\right)\left(\lim_{(x,y)\to(0,0)} \cos(y)\right)$$

Since e^x does not depend on y, $\lim_{(x,y)\to(0,0)} e^x = \lim_{x\to 0} e^x = 1$.

And since $\cos(y)$ does not depend on x, $\lim_{(x,y)\to(0,0)} \cos(y) = \lim_{y\to 0} \cos(y) = 1$.

Since both of these limits exist, the product law allows us to conclude that $\lim_{(x,y)\to(0,0)} e^x \cos(y) = \left(\lim_{(x,y)\to(0,0)} e^x\right)\left(\lim_{(x,y)\to(0,0)} \cos(y)\right) = 1$. A similar argument gives that $\lim_{(x,y)\to(0,0)} e^x \sin(y) = 0$. And then the definition of the limit gives us that:

$$\lim_{z\to 0} e^z = 1 + 0i = 1$$

We've seen a couple of examples of working out limits by hand. In practice, this is a pain. Even in your first year course in calculus, you had tools for working with limits. These same tools, namely the limit laws and continuity, are still applicable in $\mathbb{C}$.

Theorem 2.1.1: The Limit Laws

Let $f, g : U \to \mathbb{C}$ and $z_0 \in U$. If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, then:

- $\lim_{z \to z_0} \omega = \omega$ for any $\omega \in \mathbb{C}$
- $\lim_{z \to z_0} \omega f(z) = \omega A$ for any $\omega \in \mathbb{C}$
- $\lim_{z \to z_0} (f + g)(z) = A + B$
- $\lim_{z \to z_0} (fg)(z) = AB$
- If $B \neq 0$, then $\lim_{z \to z_0} \frac{f}{g}(z) = \frac{A}{B}$.

You have a lot of practice using these already. Their application is identical to their use over $\mathbb{R}$. Also, the proofs of these facts are identical to the proofs over $\mathbb{R}$, so we will not reproduce them here.

2.2 Continuity

By far the most useful way for finding limits is continuity. If a function is continuous, finding limits for it becomes immediate. So knowing what continuity gives us, and then building up a repetoire of continuous functions, is really important.

Definition 2.2.1: Continuity

Let $f : U \to \mathbb{C}$. We say that f is continuous at $z_0 \in U$ if $\lim_{z \to z_0} f(z) = f(z_0)$. f is called continuous if it is continuous on its domain.

So, if we know a function is continuous, then finding limits turns into just evaluating your function at that point.

Example 2.2.1

The function $f(z) = e^z$ is continuous.

While this seems like it should be true, we still need to show it. However, if we reproduce our

argument for showing that $\lim_{z\to 0} e^z = 1$, we find:

$$\begin{aligned}
\lim_{z\to x_0+iy_0} e^z &= \lim_{(x,y)\to(x_0,y_0)} e^x(\cos(y)+i\sin(y)) \\
&= \left(\lim_{x\to x_0} e^x\right)\left(\lim_{y\to y_0}\cos(y)+i\sin(y)\right) \\
&= e^{x_0}\left(\lim_{y\to y_0}\cos(y)+i\lim_{y\to y_0}\sin(y)\right) \\
&= e^{x_0}(\cos(y_0)+i\sin(y_0)) \\
&= e^{z_0}
\end{aligned}$$

Exercise

Prove that $f(z) = z$ is continuous.

We have a few basic functions, but what about combining them? The limit laws tell us that continuous functions combine very nicely.

Theorem 2.2.1: Properties of Continuous Functions

Let f, g be functions continuous at z_0, and h continuous at $f(z_0)$. Then:

- Constants are continuous.
- Constant multiples of f are continuous at z_0.
- Sums, differences, and products of f and g are continuous at z_0.
- If $g(z_0) \neq 0$, then $\frac{f}{g}$ is continuous at z_0.
- $h \circ f$ is continuous at z_0.

As a result of this, most of the functions we've seen so far are continuous. Constants, polynomials, exponentials, and our trig functions are continuous.

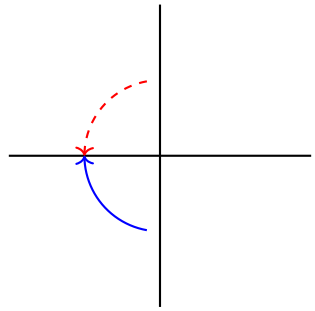
What about logarithms, or the argument?

Example 2.2.2

Let $\arg_0(z)$ be the branch of the argument defined by setting $\arg_0(z) \in [-\pi, \pi)$.

Find $\lim_{z\to -1} \arg_0(z)$.

Since this limit needs to exist and agree regardless of which direction we approach -1 from, we're going to approach along two curves: we're going to follow the unit circle to -1 from above and from below.



First, let's discuss what happens if we follow the curve $z = e^{i\theta}$ as it approaches -1 from above (i.e., as we follow the red, dashed curve.) Notice that since we are in the second quadrant, that $\arg_0(z) \in \left(\frac{\pi}{2}, \pi\right]$, and so we see that:

$$\theta \to \pi^-$$

As such, $\lim_{z\to -1} \arg_0(z) = \lim_{\theta\to\pi^-} \theta = \pi$.

On the other hand, if we approach -1 from below along the blue, solid curve, we see that $\arg_0(z) \in \left(-\pi, \frac{-\pi}{2}\right)$, and so:

$$\theta \to -\pi^+$$

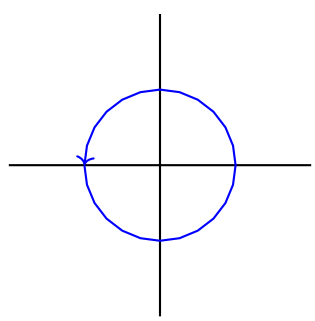
And so, in a similar way to the previous curve, we find that $\lim_{z\to -1} \arg_0(z) = -\pi$.

Since the limit approaches two different values, it does not exist. Also, as a consequence, we see that $\arg_0(z)$ is not continuous at -1.

Note. A similar argument will show that $\arg_0$ and $\log_0$ are not continuous on $(-\infty, 0]$, but that they are continuous on $\mathbb{C} \setminus (-\infty, 0]$.

This is why we defined $\text{Arg}(z)$ and $\text{Log}(z)$ to have the domain $\mathbb{C}\setminus(-\infty, 0]$. We want to work with continuous (actually, differentiable) functions, and so we define these functions on a set where they are continuous.

The issue here is that $\arg(z)$ and $\log(z)$ are multivalued functions. As we move around the circle $|z| = 1$:

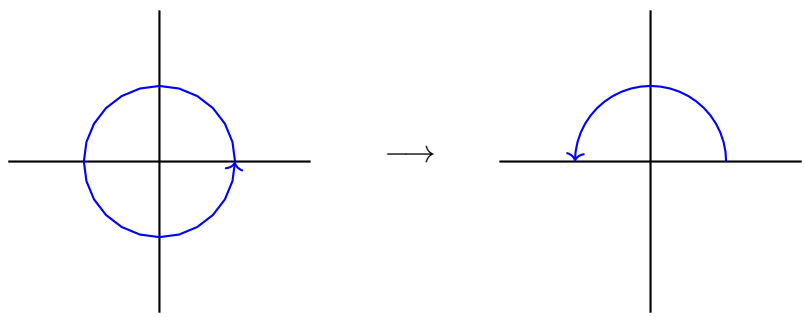


the value of $\arg(z)$ increases by 2π, and the value of $\log(z)$ increases by $2\pi i$. The same thing happens with other multivalued functions.

Example 2.2.3

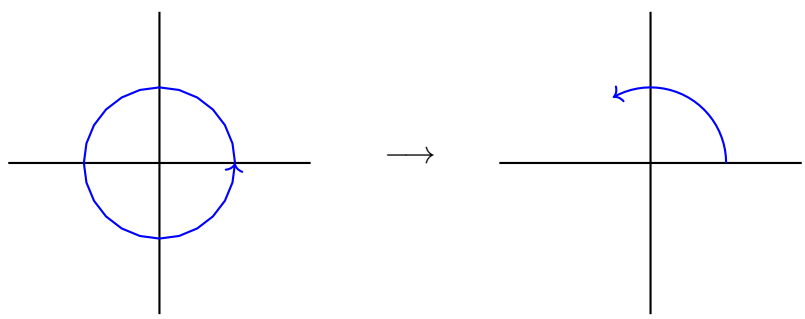
What happens to $z^{\frac{1}{2}}$ and $z^{\frac{1}{3}}$ if we travel around the unit circle?

For $z^{\frac{1}{2}}$, we use the formula $\sqrt{r}e^{i\frac{\theta}{2}}$. So, starting at 1 written as e^{i0}, we have $1^{\frac{1}{2}} = 1$. As we move around the circle once (clockwise), we see that θ moves from 0 to 2π, and so $z^{\frac{1}{2}}$ moves from $e^{i0} = 1$ to $e^{i\pi} = -1$. Pictorially:



And to get back to 1, we need to go around the unit circle twice!

A similar line of reasoning applies to the multi-valued function $z^{\frac{1}{3}}$. In this case, the picture looks like:



In this case, as we travel the unit circle once, $z^{\frac{1}{3}}$ goes from 1 to $\omega_1 = e^{i\frac{2\pi}{3}}$, which is another cube root of unity.

So what's the solution to this problem? We want to work with continuous functions, if we're going to talk about differentiation. We've seen an example already: for $\log(z)$, we restricted the domain to $\mathbb{C} \setminus (-\infty, 0]$, and we were able to find a branch on that domain which is continuous.

This leads us to a more general phenomenon.

2.2.1 Branch Cuts

The idea behind branch cuts is to find a way to take discontinuous branches of a multi-valued function, and to remove a portion of the domain to get a continuous function.

Definition 2.2.2: Branch Cuts

Let $f(z)$ be a branch of a multi-valued function. A branch cut is a curve in $\mathbb{C}$ along which $f(z)$ is discontinuous.

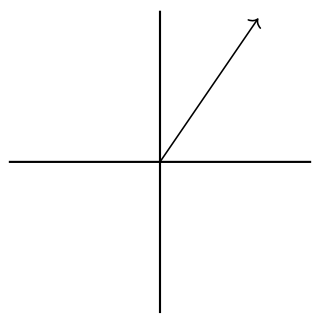
It is called a branch **cut** because we cut (i.e. remove) the branch cut from the domain to get a continuous function.

Example 2.2.4

Let $\log_0(z)$ be the branch of the logarithm given by $\arg(z) \in [-\pi, \pi)$. As we saw earlier, $\log_0(z)$ is discontinuous along $(-\infty, 0]$, and so this is a branch cut. Removing $(-\infty, 0]$ from the domain of $\log_0(z)$ results in the function $\text{Log}(z)$, which is continuous on its domain.

All of the multi-valued functions we have seen so far are defined in terms of $\log(z)$ or $\arg(z)$. As such, we can get branch cuts for them by taking branch cuts for $\log(z)$ or $\arg(z)$. Let's start by talking about how to do that.

Consider the branch $\log_0(z)$ of $\log(z)$ given by $\arg(z) \in (\theta, \theta + 2\pi)$ for some $\theta \in \mathbb{R}$. This function is continuous on its domain, in the same way that $\text{Log}(z)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$. We have removed the ray $\{re^{i\theta} | r \geq 0\}$, pictured below:



By taking this branch cut, we have found a continuous branch. In practice, these are the only types of branch cuts we will consider.

Now that we have a standard way of taking branch cuts for $\log(z)$ or $\arg(z)$, how do these choices affect other multivalued functions?

Example 2.2.5

Consider $f(z) = (iz + 1)^{\frac{1}{2}}$, where we are working with the principal branch. What is the branch cut of this function?

Since we are asking about a branch cut, this should serve as a huge clue that this function involves $\log(z)$ somehow. Recall that $(iz + 1)^{\frac{1}{2}} = e^{\frac{1}{2}\text{Log}(iz+1)}$.

Now, our branch cut on $\text{Log}(z)$ is to remove $(-\infty, 0]$ from the domain. So the corresponding branch cut for $f(z)$ is to remove where $iz + 1 \in (-\infty, 0]$.

Suppose $iz + 1 \in (-\infty, 0]$. Then $iz \in (-\infty, -1]$. As such, $z \in \left\{\frac{r}{i} | r \leq -1\right\} = \{si | s \geq 1\}$. As such, our branch cut is the positive imaginary axis above 1.

2.3 Infinite Limits and Limits at Infinity

The last thing to consider is infinite limits and limits at infinity. The key concept here is that in $\mathbb{C}$, there is only one ∞. No matter which direction you go out in, left or right, up or down, you get to the same infinity. This is tied to the notion of the Riemann sphere, and of Riemann surfaces, which are a geometric abstraction that makes a lot of complex analysis very nice. We won't be talking in this context in this course. I am pointing these out in case you are interested in further reading.

So, how do we define infinite limits? Well, $f(z)$ should be going to ∞. But what way? Well, since all directions give the same ∞, any way!

Definition 2.3.1: Infinite Limits

Let $f : U \to \mathbb{C}$ and $z_0 \in \mathbb{C}$, such that $\{z \in \mathbb{C} | 0 < |z - z_0| < r\} \subset U$ for some $r > 0$. Then we say that $\lim_{z \to z_0} f(z) = \infty$ if:

$$\forall N > 0, \exists \delta > 0 \text{ such that } 0 < |z - z_0| < \delta \implies |f(z)| > N$$

In laymans terms, this means that $|f(z)|$ gets arbitrarily large as we get close to z_0.

Example 2.3.1

Is $\lim_{z \to 0} e^{\frac{1}{z}}$ infinite or not?

Well, for example, if we consider $z \to 0$ along the positive real axis, we find that $e^{\frac{1}{x}}$ does go to infinity. After all, $\lim_{x \to 0^+} \frac{1}{x} = \infty$.

However, consider what occurs as $z \to 0$ along the positive imaginary axis. We have that $e^{\frac{1}{z}} = e^{\frac{1}{ri}} = e^{\frac{-i}{r}}$. And so $|e^{\frac{1}{z}}| = 1$. As such, if we approach from this direction, $e^{\frac{1}{z}}$ does not approach ∞.

So this limit is not infinite.

What about limits at infinity?

Definition 2.3.2: Limits at Infinity

Let $f : U \to \mathbb{C}$ such that $\{z \in \mathbb{C} | |z| > r\} \subset U$ for some $r > 0$. Then we say that $\lim_{z \to \infty} f(z) = L$ if:

$$\forall \varepsilon > 0, \exists M > 0 \text{ such that } |z| > M \implies |f(z) - L| < \varepsilon$$

Or, in laymans terms, as $|z|$ gets arbitrarily large, $f(z)$ gets arbitrarily close to L.

Example 2.3.2

Show that $\lim_{z \to \infty} z = \infty$.

So, we haven't defined this, but hopefully seeing the two separate definitions allows us to synthesize the correct definition of an infinite limit at infinity.

In this case, we want that as $|z|$ gets arbitrarily large, that $|f(z)| = |z|$ gets arbitrarily large, which

is obviously the case.

2.4 The Complex Derivative

As I have repeatedly mentioned, our goal is to discuss calculus over $\mathbb{C}$. We can finally begin to talk about this! We start with the derivative, which is defined exactly how you would expect:

Definition 2.4.1: The Derivative

Let $f : U \to \mathbb{C}$, $z_0 \in U$, and $\{z \in \mathbb{C} | |z - z_0| < r\} \subset U$ for some $r > 0$. We say that $f(z)$ is differentiable at z_0, with derivative $f'(z_0)$ if:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad \text{exists}$$

Before we begin to dive into the theory, let's work out an example.

Example 2.4.1

Let $f(z) = iz^2 + 2z$. Find $f'(z)$ from the definition.

$$\begin{aligned} f'(z) &= \lim_{h \to 0} \frac{i(z+h)^2 + 2(z+h) - (iz^2 + 2z)}{h} \\ &= \lim_{h \to 0} \frac{i(z^2 + 2zh + h^2) + 2z + 2h - iz^2 - 2z}{h} \\ &= \lim_{h \to 0} \frac{2izh + h^2 + 2h}{h} \\ &= 2iz + 2 \end{aligned}$$

So $f'(z) = 2iz + 2$, which is what we were expecting.

Because complex limits are really limits in 2 dimensions, we need to be careful. For example, if you were going to try fo find the derivative of e^z by definition, you would run into some very nasty $\mathbb{R}^2$ limits. We don't want that in our lives.

However, the solution to avoiding working with nasty $\mathbb{R}^2$ limits is to leverage the 2-dimensional nature of the derivative!

Theorem 2.4.1

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at z_0. Then the following two equations hold:

$$f'(z) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

$$f'(z) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0)$$

Proof. Since $f'(z_0)$ exists, we know that $lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$ exists. As such, it must exist from every direction. We will consider approaching along two lines: $h = a + 0i$ and $h = 0 + ib$. I.e., along the real and imaginary axes.

Along the real axis, we get the limit:

$$\begin{aligned}
f'(z_0) &= \lim_{a\to 0}\frac{f(z_0+a)-f(z_0)}{a} \\
&= \lim_{a\to 0}\frac{u(x_0+a,y_0)+iv(x_0+a,y_0)-u(x_0,y_0)-iv(x_0,y_0)}{a} \\
&= \lim_{a\to 0}\frac{u(x_0+a,y_0)-u(x_0,y_0)}{a} + i\lim_{a\to 0}\frac{v(x_0+a,y_0)-v(x_0,y_0)}{a} \\
&= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)
\end{aligned}$$

And along the imaginary axis:

$$\begin{aligned}
f'(z_0) &= \lim_{b\to 0}\frac{f(z_0+ib)-f(z_0)}{ib} \\
&= -i\lim_{b\to 0}\frac{u(x_0,y_0+b)+iv(x_0,y_0+b)-u(x_0,y_0)-iv(x_0,y_0)}{b} \\
&= -i\left(\lim_{b\to 0}\frac{u(x_0,y_0+b)-u(x_0,y_0)}{b} + i\lim_{b\to 0}\frac{v(x_0,y_0+b)-v(x_0,y_0)}{b}\right) \\
&= -i\left(\frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0)\right) \\
&= \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0)
\end{aligned}$$

We therefore have the two expressions for $f'(z_0)$ that we desired. □

Notice, this gives two expressions for the derivative! Since the limit has only one value, we an conclude that these two expressions are actually equal.

Corollary 2.4.1: The Cauchy-Riemann Equations

If $f(z)$ is differentiable at z_0, then $f(z)$ satisfies the **Cauchy-Riemann** equations at z_0. Namely:

$$u_x = v_y$$

$$u_y = -v_x$$

Let's see a couple of examples for how we can use these two results.

Example 2.4.2

The principal logarithm $\text{Log}(z)$ is differentiable on $\mathbb{C}\setminus(-\infty,0]$. We will justify why this is true a bit later, but for now we can accept it. Prove that $\text{Log}(z)' = \frac{1}{z}$ where $\text{Re}(z) > 0$.

Let $z = x + iy$. Then $\text{Log}(z) = \ln|z| + i\text{Arg}(z)$. So:

$$u(x,y) = \ln\sqrt{x^2+y^2}$$

$$v(x,y) = \arctan\left(\frac{y}{x}\right)$$

We know that $f'(z) = u_x + iv_x$. We compute:

$$\begin{aligned} u_x &= \frac{x}{\sqrt{x^2+y^2}^2} \\ v_x &= \frac{1}{\left(\frac{y}{x}\right)^2+1}\left(\frac{-y}{x^2}\right) \\ &= \frac{-y}{\left(\frac{y}{x}\right)^2+1}\frac{1}{x^2} \\ &= \frac{-y}{x^2+y^2} \end{aligned}$$

As such, the derivative is $\frac{x-iy}{x^2+y^2} = \frac{\overline{z}}{|z|^2} = \frac{1}{z}$.

Example 2.4.3

Find where $f(z) = \cos(x)$ is differentiable.

We have that $u(x,y) = \cos(x)$ and $v(x,y) = 0$. If this were differentiable, it would satisfy the Cauchy-Riemann equations. We would have:

$$\sin(x) = u_x = v_y = 0$$

$$0 = u_y = -v_x = 0$$

So, this function does not satisfy the Cauchy-Riemann equations when $\sin(x) \neq 0$. I.e, when $x \neq k\pi$.

What happens when $x = k\pi$? Consider:

$$f'(k\pi) = \lim_{a+bi\to 0}\frac{\cos(k\pi+a)-1}{a+bi} = \lim_{a+bi\to 0}\frac{(-1)^k\cos(a)-(-1)^k}{a+bi}$$

Now, since $|a+ib| \geq |a|$, we know that $\left|\frac{(-1)^k\cos(a)-(-1)^k}{a+ib}\right| \leq \left|\frac{(-1)^k\cos(a)-(-1)^k}{a}\right|$. Therefore:

$$\lim_{(a,b)\to(0,0)}\frac{(-1)^k\cos(a)-(-1)^k}{a} = (-1)^k\lim_{a\to 0}\frac{\cos(a)-1}{a} = 0$$

by L'Hopital (in $\mathbb{R}$). As such, the squeeze theorem tells us that $\lim_{h\to 0}\frac{\cos(k\pi+a)-(-1)^k}{a+ib} = 0$.

So $\cos(x)$ is differentiable at exactly the points $z = k\pi$.

In this example, we saw an example of a function that was differentiable at exactly the places where it satisfied the Cauchy-Riemann equations. Is that generally true? That would make life very nice for us.

Example 2.4.4

Consider $f(z) = \sqrt{|xy|}$. Prove that this satisfies the Cauchy-Riemann equations at 0, but is not differentiable there.

Since $\sqrt{|xy|}$ is real, $u(x, y) = \sqrt{|xy|}$ and $v(x, y) = 0$. Now, computing the partial derivatives using differentiation rules will not work here. We need to use the defintion:

$$u_x(0, 0) = \lim_{a \to 0} \frac{\sqrt{|a \times 0|} - \sqrt{|0 \times 0|}}{x} = 0$$

And similarly, $u_y = 0$. As such, $u_x = 0 = v_y$ and $u_y = 0 = -v_x$. So $f(z)$ satisfies the Cauchy-Riemann equations at $z = 0$.

However, we need:

$$\lim_{(a,b) \to (0,0)} \frac{\sqrt{|ab|}}{a + ib}$$

to exist. That means it must exist along every direction! We have shown that it is 0 along the real and imaginary axes. Consider the direction where $x = y$, and $x > 0$. We get:

$$\lim_{(a,a) \to (0^+,0^+)} \frac{\sqrt{|a^2|}}{a + ia} = \lim_{a \to 0^+} \frac{a}{a + ia} = \frac{1}{1 + i} \neq 0$$

Since we get different limits along different directions, $f'(0)$ does not exist. Therefore, f satisfies the Cauchy-Riemann equations at $z = 0$, but is not differentiable there!

We would like to have an easy to check, general condition to see if a function is differentiable. The Cauchy-Riemann equations seemed like a good bet, but we've seen that they aren't sufficient to guarantee that a function is differentiable. Can we fix this?

The answer is yes. There is an easy condition we can add, which will salvage the usefulness of the Cauchy-Riemann equations. However, to properly discuss it we will need to take a brief detour into topology.

2.4.1 The Topology of $\mathbb{C}$

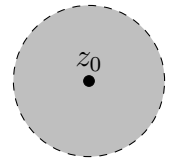
Topology is a fairly broad field. Fortunately for us, we will only need some basic definitions. At its most basic, topology is concerned with the notion of an "open set". For $\mathbb{C}$, these turn out to be fairly nice. They're based on open balls:

Definition 2.4.2: Open Ball

Let $z_0 \in \mathbb{C}$. An open ball of radius $r > 0$ centered at z_0 is a set:

$$B_r(z_0) = \{z \in \mathbb{C} | |z - z_0| < r\}$$

So they're just filled in circles missing their boundaries:



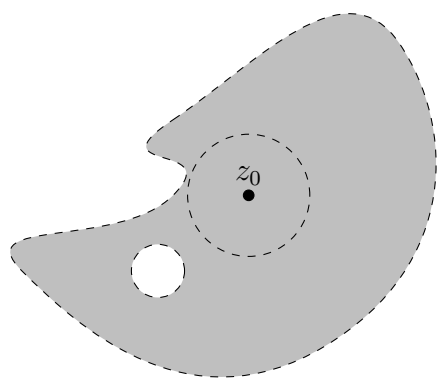
Open balls are the basic building blocks of open sets. There are two equivalent characterisations:

Definition 2.4.3: Open Set

We say that a subset $U \subset \mathbb{C}$ is open if for any $z_0 \in U$, there exists a ball $B_r(z_0)$ which is contained in U.

Alternatively, an open set is an arbitrary union of open balls.

We can visualize this definition as:



Notice that around the point z_0, we have found a ball $B_r(z_0)$ contained entirely in U, which is the shaded region. We could do the same for any point in the shaded region, if we desired.

How do we recognize open sets in practice? If we have a picture, it's easy to do so. The set shouldn't contain any "edge" or boundary. This is an intuitive notion which is easy to see visually, so we won't go through the effort of describing it formally. There is a formal definition, but it isn't very helpful for developing a visual heuristic.

Algebraically, it's much easier to determine open sets. Open sets are generally described by conditions involving inequalities of some variety. For example: $\{z \in \mathbb{C} | \operatorname{Re}(z) > 1\}$ is an open set. Open balls are open sets. $\mathbb{C}$ and the empty set are open. And there are many other examples.

In addition to open sets, we need one more topological notion: connectedness. The idea behind a set being "connected" is that it is in one piece. To describe this formally, we need to talk about paths.

Definition 2.4.4: path

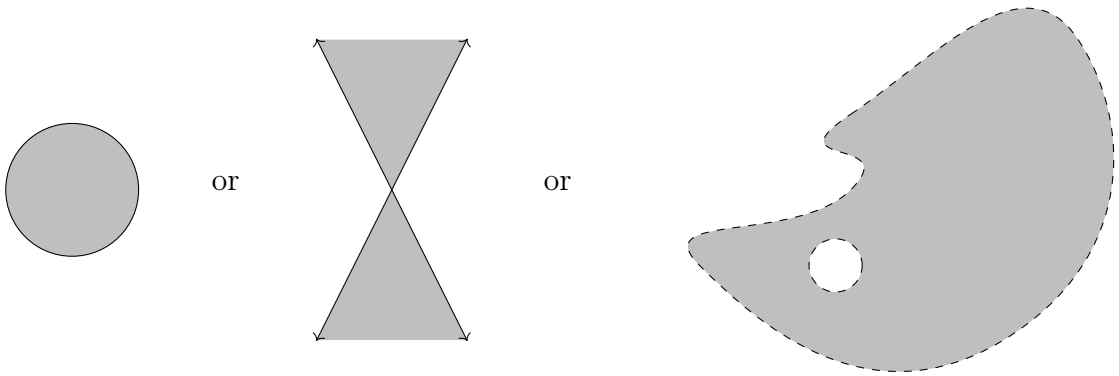
A path γ in $\mathbb{C}$ is a function $\gamma : [0, 1] \to \mathbb{C}$ such that $\operatorname{Re}(\gamma)$ and $\operatorname{Im}(\gamma)$ are continuous functions.

In a set that is in one piece, it should be possible to draw a path between two points in the set without leaving the set. And this is precisely the definition of connectedness.

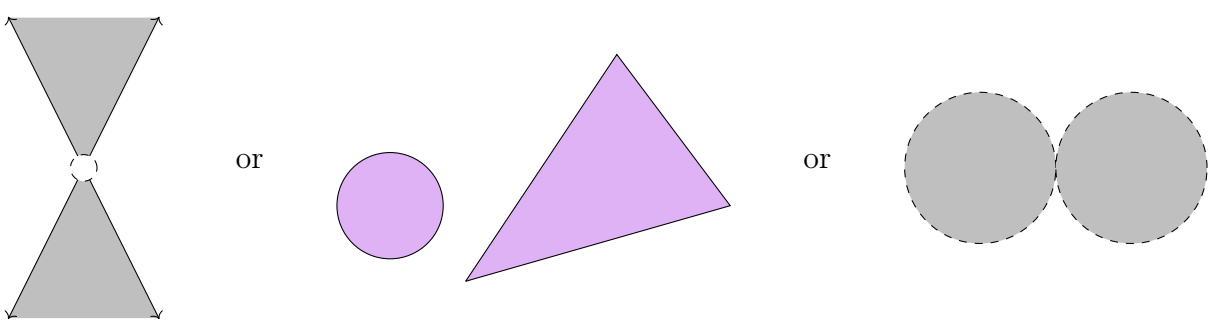
Definition 2.4.5: Connected Set

A set $U \subset \mathbb{C}$ is called connected if for any two $z_0, z_1 \in U$ there exists a path γ such that $\gamma(0) = z_0$, $\gamma(1) = z_1$, and $\gamma(t) \in U$ for all t.

Some examples of connected sets:



On the other hand, these are sets that are not connected:



The first of these is hopefully self-explanatory. It's in two pieces. The second of these, which consists of the triangle and the circle, is also in two discrete pieces.

In the third picture, we have open balls which are tangent to one another. However, since they are missing their boundaries, these two sets actually have a gap between them!

Putting these two notions together, we have the basic topological object we will be dealing with in this course:

Definition 2.4.6: Domain

A set $U \subset \mathbb{C}$ is called a domain if it is open and connected.

Note. This is not the same thing as the domain of a function! You need to be able to tell the difference from context. If we are not discussing a function, then domain will be referring to this definition. In contexts where

a function is also being discussed, you will need to be vigilant. There is a difference between the language "the domain U" and "the domain U of f."

Note. The last note, while an important technical issue, doesn't quite give the full picture.

In the context of this course, domains will almost only ever occur as the domain of a differentiable function. (In fact, what's called an "analytic function", which we will see soon.)

2.4.2 Returning to Differentiation

Now that we have some topological language, we can get to the point: how do we tell if a function is differentiable?

Definition 2.4.7: Holomorphic or Analytic Functions

A function f is holomorphic (or analytic) on an open set U if it is differentiable at each point in U.
A function f is called holomorphic (or analytic) if it is holomorphic on its domain.

Note. The textbook does not use the terminology "holomorphic" to describe these functions. However, to be precise, analytic really means something different. An analytic function is a function that is equal to a power series.

However, it is one of the crowning achievements of complex analysis that every holomorphic function is analytic. (I.e., if f is differentiable on an open set, then it can be described by power series on that open set.) As such, holomorphic and analytic are the same condition on $\mathbb{C}$. This is not true on $\mathbb{R}$ that differentiable and analytic are the same though, so it is worthwhile to make this distinction.

Plus, I just like the sound of holomorphic more.

So what was the point of introducing all these definitions? Do we get something out of this?

Theorem 2.4.2

Suppose $f = u + iv$ is defined on an open set U. If u, v, u_x, u_y, v_x, v_y are defined and continuous everywhere in U and u, v satisfy the Cauchy-Riemann equations, then f is holomorphic on U.

Proof. The proof of this requires a result from multivariable calculus:

If u_x and u_y are continuous, then u is differentiable. This means that there exists a function $\varepsilon_u(x, y)$ such that:

$$u(x + a, y + b) = u(x, y) + au_x + bu_y + \varepsilon_u(a, b)$$

The function ε_u satisfies the condition that:

$$\lim_{(a,b)\to(0,0)} \frac{\varepsilon_u(a, b)}{\sqrt{a^2 + b^2}} = 0$$

And similarly:

$$v(x + a, y + b) = v(x, y) + av_x + bv_y + \varepsilon_v(a, b)$$

where ε_v is defined similarly for v.

As such:

$$\begin{aligned}
\lim_{a+ib\to 0}\frac{f(z+h)-f(z)}{h} &= \lim_{a+ib\to 0}\frac{u(x+a,y+b)+iv(x+a,y+b)-(u(x,y)+iv(x,y))}{a+ib}\\
&= \lim_{a+ib\to 0}\frac{au_x+bu_y+i(av_x+bv_y)+\varepsilon_u(a,b)+i\varepsilon_v(a,b)}{a+ib}\\
&= \lim_{a+ib\to 0}\frac{au_x-bv_x+i(av_x+bu_x)}{a+ib}+\lim_{a+ib\to 0}\frac{\varepsilon_u(a,b)+i\varepsilon_v(a,b)}{a+ib}
\end{aligned}$$

Now, $au_x - bv_x + i(av_x + bu_x) = (a+ib)u_x + (ia-b)v_x = (a+ib)u_x + i(a+ib)v_x$. As such:

$$f'(z) = u_x + iv_x + \lim_{a+ib\to 0}\frac{\varepsilon_u(a,b)+i\varepsilon_v(a,b)}{a+ib}$$

So we need only consider this last limit. However, note that by the triangle inequality:

$$\left|\frac{\varepsilon_u(a,b)+i\varepsilon_v(a,b)}{a+ib}\right| \leq \frac{|\varepsilon_u(a,b)|}{\sqrt{a^2+b^2}}+\frac{|\varepsilon_v(a,b)|}{\sqrt{a^2+b^2}}$$

By the definition of ε_u and ε_v, we know that:

$$\lim_{(a,b)\to(0,0)}\frac{|\varepsilon_u(a,b)|}{\sqrt{a^2+b^2}}+\frac{|\varepsilon_v(a,b)|}{\sqrt{a^2+b^2}} = 0$$

So by the squeeze theorem, $\lim_{a+ib\to 0}\frac{\varepsilon_u(a,b)+i\varepsilon_v(a,b)}{a+ib} = 0$ as well.

As such, $f'(z) = u_x + iv_x$, and f is differentiable at z. Since this applies for any $z \in U$, we see that f is holomorphic on U. □

Example 2.4.5

Show that e^z is analytic on $\mathbb{C}$.

We need to write e^z as $u+iv$, and show that u, v, u_x, u_y, v_x, v_y are continuous. We also need to show that u, v satisfy the Cauchy-Riemann equations.

To start, $e^z = e^x e^{iy} = e^x\cos(y) + ie^x\sin(y)$. As such, $u(x,y) = e^x\cos(y)$ and $v(x,y) = e^x\sin(y)$. These are continuous.

We compute the partials:

$$u_x = e^x\cos(y)$$

$$u_y = -e^x\sin(y)$$

$$v_x = e^x\sin(y)$$

$$v_y = e^x\cos(y)$$

Notice that these are all continuous. And further, that:

$$u_x = e^x \cos(y) = v_y$$

$$u_y = -e^x \sin(y) = -v_x$$

As such, e^z satisfies the conditions of the theorem, and is therefore analytic on $\mathbb{C}$.

Functions which are analytic on $\mathbb{C}$ are special. They have some very nice properties. For example, their range is always either $\mathbb{C}$ or $\mathbb{C} \setminus \{z_0\}$ for some $z_0 \in \mathbb{C}$. This is called Picard's Little Theorem. We won't be using that in this course, but it's a good example of why these functions are special. As such, they have a name:

Definition 2.4.8: Entire Function

A function which is analytic on $\mathbb{C}$ is called **entire**.

We'll talk more about entire functions as the course moves on.

How do we compute derivatives in practice? After all, computing partials and then turning them into a function of z can be quite nasty. Is there a better way?

Theorem 2.4.3: The Derivative Rules

Suppose f, g are differentiable at z_0 and h is differentiable at $f(z_0)$. Then:

- The constant multiple rule: $\omega f(z_0) = \omega f'(z_0)$ for any $\omega \in \mathbb{C}$.
- The sum rule: $(f+g)'(z_0) = f'(z_0) + g'(z_0)$.
- The product rule: $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$.
- The quotient rule: if $g(z_0) \neq 0$, then $\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$.
- The chain rule: $(h \circ f)'(z_0) = h'(f(z_0))f'(z_0)$.

Combining this with the basic derivatives we know: z, e^z, $\mathrm{Log}(z)$, etc. allows us to figure out the derivatives of basically any function we want.

Example 2.4.6

Find the derivative of z^n.

Well, we could us the definition of the derivative in combination with the binomial theorem.

Instead, let's use the product rule and induction. We claim that if $f(z) = z^n$, then $f'(z) = nz^{n-1}$ for integers $n \geq 0$.

Base case: When $n = 0$, we know that $f(z) = 1$, so $f'(z) = 0$. After all, $\frac{f(z+h)-f(z)}{h} = \frac{1-1}{h} = 0$ for $h \neq 0$. So the formula holds when $n = 0$.

For $n = 1$, we have $\lim_{h\to 0} \frac{(z+h)-z}{h} = \lim_{h\to 0} \frac{h}{h} = 1$. So the formula also holds for $n = 1$.

Induction hypothesis: Suppose the derivative of z^n is nz^{n-1} for some $n \geq 0$.

Induction step: Consider z^{n+1}. Let $f(z) = z^n$ and $g(z) = z$. Then, by the product rule:

$$\frac{d\,z^{n+1}}{d\,z} = f'(z)g(z) + f(z)g'(z) = (nz^{n-1})z + z^n(1) = nz^n + z^n = (n+1)z^n = (n+1)z^{(n+1)-1}$$

As such, the formula holds for $n + 1$ as well. By induction, our claim holds for all integers $n \geq 0$.

What about non-integer powers? First, we need to find the derivative of any branch of $\log(z)$.

Example 2.4.7

Suppose $\log_0(z)$ is a continuous branch of $\log(z)$, given by taking $\arg(z) \in (\theta, \theta + 2\pi)$. We have already shown that $\mathrm{Log}(z)$ is differentiable. A similar argument will work here. So we'll move ahead with finding the derivative.

We know that $e^{\log_0(z)} = z$. By the chain rule:

$$1 = \frac{d\,z}{d\,z} = \frac{d\,e^{\log_0(z)}}{d\,z} = e^{\log_0(z)}\frac{d\,\log_0(z)}{d\,z}$$

So, $1 = z\frac{d\,\log_0(z)}{d\,z}$. So the derivative of $\log_0(z)$ is $\frac{1}{z}$.

Example 2.4.8

Find the derivative of any branch of z^α where $\alpha \in \mathbb{C}$.

By definition, $f(z) = z^\alpha = e^{\alpha \log_0(z)}$, where $\log_0(z)$ is the branch of the logarithm corresponding to the branch of z^α.

By the chain rule, $f'(z) = e^{\alpha \log_0(z)}(\alpha \log_0(z))' = \frac{\alpha e^{\alpha \log_0(z)}}{z}$. Can we simplify this at all?

$$\begin{aligned}\frac{\alpha e^{\alpha \log_0(z)}}{z} &= \frac{\alpha e^{\alpha \log_0(z)}}{e^{\log_0(z)}} \\ &= \alpha e^{\alpha \log_0(z) - \log_0(z)} \\ &= \alpha e^{(\alpha-1)\log_0(z)} \\ &= \alpha z^{\alpha-1}\end{aligned}$$

So, the formula we expect works not just for integers, but for any complex exponent.

2.5 Harmonic Functions

Before we move on, we have one more fact about analytic functions that we're going to discuss. It turns out, they're very closely related to harmonic functions.

Definition 2.5.1: Harmonic Function

Let U be an open set in $\mathbb{C}$. A function $u : U \to \mathbb{R}$ is harmonic if it satisfies that:

$$u_{xx} + u_{yy} = 0$$

This is sometimes also written as: $\Delta u = 0$ or $\nabla^2 \cdot u = 0$. This is called Laplace's equation.

Now, it is a very nice fact about holomorphic functions $f = u + iv$ that u and v are actually second differentiable, and u_{xy}, u_{yx}, v_{xy}, and v_{yx} are all continuous. In fact, we will see later that if f is holomorphic, then f is infinitely differentiable. For now, we will take this for granted.

Theorem 2.5.1

Suppose f is an analytic function on U. Then u and v are harmonic functions on U.

Proof. By our discussion above, we know that we can apply Clairaut's theorem to get:

$$\begin{aligned}
u_{xx} &= \frac{\partial u_x}{\partial x} \\
&= \frac{\partial v_y}{\partial x} && \text{(by C-R)} \\
&= v_{xy} = v_{yx} && \text{(by Clairaut)} \\
&= \frac{\partial v_x}{\partial y} = \frac{\partial(-u_y)}{\partial y} && \text{(by C-R)} \\
&= -u_{yy}
\end{aligned}$$

So $u_{xx} + u_{yy} = -u_{yy} + u_{yy} = 0$, and u is harmonic. A similar argument shows that v is also harmonic. $\square$

Example 2.5.1

Does there exist a holomorphic function $f = u + iv$ so that $u(x, y) = x^2$?

This theorem tells us that if such a function f exists, then $u(x, y) = x^2$ must be harmonic. However:

$$u_{xx} = 2$$

$$u_{yy} = 0$$

So $\Delta u = 2 \neq 0$. Since u is not harmonic, no such f exists.

Alright, so the real (and imaginary) parts of an analytic function are harmonic. Is the converse true? I.e.,

does every harmonic function appear as the real or imaginary part of an analytic function? It turns out that the answer is yes! Let's see an example.

Example 2.5.2

Let $u(x, y) = 3x^2y - y^3$. Find an analytic function $f(z)$ whose real part is $u(x, y)$.

First, notice that u is harmonic. We won't use this explicitly anywhere, but our goal is to demonstrate that harmonic functions do appear as the real parts of an analytic function.

If $f(z)$ is such a function, it must satisfy the Cauchy-Riemann equations. Let $f(z) = u + iv$. Then:

$$v_y = u_x = 6xy$$

As such, we can see that $v(x, y) = \int u_x dy = 3xy^2 + g(x)$ for some function $g(x)$. Why do we get this $g(x)$? Well, we're integrating in terms of y. That only recovers y information. In particular, it misses any parts of v that depend only on x! So we need to assume there is some part of v that depends only on x, and then try to determine what that is.

To do so, let's look at v_x. We have $v_x = 3y^2 + g'(x)$. However, by Cauchy-Riemann, we know that:

$$3y^2 + g'(x) = v_x = -u_y = -3x^2 + 3y^2$$

As such, $g'(x) = -3x^2$, and so $g(x) = -x^3 + C$ for some constant C.

So, we find that $f(z) = u+iv = (3x^2y-y^3)+i(3xy^2-x^3+C) = i(-x^3-i3x^2y+3xy^2+iy^3)+iC$. With some fiddling, one can notice that $x^3 + i3x^2y - 3xy^2 - iy^3 = (x + iy)^3 = z^3$. As such, $f(z) = -iz^3 + iC$ for some $C \in \mathbb{R}$.

This example turns out to be archetypical. Given $u(x, y)$, finding $v(x, y)$ so that $u + iv$ is analytic boils down to solving a system of partial differential equations. However, the procedure isn't unreasonable. Such u and v are called harmonic conjugates:

Definition 2.5.2: Harmonic Conjugates

Let $u(x, y)$ be a harmonic function. We say that v is a harmonic conjugate for u if $f(z) = u + iv$ is holomorphic.

Example 2.5.3

Why do we need this notion? Isn't it true that if u, v are harmonic, then $u + iv$ is holomorphic?

Well, no. Consider $u(x, y) = x$ and $v(x, y) = -y$. Then these are both harmonic. However, $f(z) = x - iy = \overline{z}$, which we know is nowhere differentiable.

So, it is not true that if you take any two arbitrary harmonic functions u, v, that $u+iv$ is holomorphic. This is something very special.

In light of this example, it's natural to ask whether or not harmonic functions even have a harmonic conjugate. The answer is, luckily, yes.

Theorem 2.5.2

If u is harmonic on an open set U, then there exists a harmonic conjugate v for u.

Proving this amounts to solving the system of differential equations:

$$v_y = u_x$$

$$v_x = -u_y$$

Since then u, v satisfy Cauchy-Riemann and $f = u + iv$ will be holomorphic. (We are guaranteed that u, v and their partials are continuous since they are harmonic.) We know u_x and $-u_y$, so we just need to know how to solve the system $v_x = f$ and $v_y = g$. The theoretical idea is to integrate. We will eschew giving a precise proof; our topic isn't PDEs after all.

3 Integration

We are now ready to begin talking about the integration side of complex calculus. Since complex functions can be thought of as functions of two variables, we have a couple of options for integrating: surface integrals, such as $\int_U f(z)dxdy$ where we integrate over the open set U; and line integrals $\int_\gamma f(z)dz$ where we integrate over a curve γ in $\mathbb{C}$.

It turns out that surface integrals hold little interest. Line integrals are where the magic happens. Here are some facts we're going to be able to show using complex line integrals:

- Every holomorphic function is infinitely differentiable.
- Every holomorphic function is analytic, meaning it has a power series expansion around any point in the domain.
- Liouville's theorem states that any bounded entire function is constant.
- Let $a \in (0, 1)$. Then
$$\int_0^\infty \frac{x^{a-1}}{1+x}dx = \frac{\pi}{\sin(\pi a)}$$

Liouville's theorem is a neat theoretical tool, which will give us some very powerful theorems. It's going to allow us to prove the Fundamental Theorem of Algebra (every non-constant complex polynomial has roots).

Why might the last fact be interesting? Well, it's not the fact by itself that's interesting. After all, it's a seemingly random integral. (Indeed, it only holds interest to me because it appeared on my comprehensive exam for complex analysis!) What's interesting is that we can use complex line integrals to find real improper integrals.

3.1 Curves in $\mathbb{C}$

Before we can begin discussing line integrals, we should give ourselves a quick refresher on curves. After all, we need to integrate over *something*.

Definition 3.1.1: Curves

A **curve** in $\mathbb{C}$ is a function $\gamma : [a, b] \to \mathbb{C}$, for some real $a < b$, such that $\mathrm{Re}(\gamma)$ and $\mathrm{Im}(\gamma)$ are continuous.

Further, such a curve is called **smooth** if $\mathrm{Re}(\gamma)$ and $\mathrm{Im}(\gamma)$ are differentiable. It is called **piecewise smooth** if they are differentiable except at finitely many places. In this case, $\gamma' = [\mathrm{Re}(\gamma)]' + [\mathrm{Im}(\gamma)]'$.

Paths are a special case of curves, where $a = 0$ and $b = 1$.

The point $\gamma(a)$ is called the **start** of the curve, and $\gamma(b)$ is the **end** of the curve. γ is called **closed** if it starts and ends at the same point.

We are going to be using a lot of different paths throughout this course. We present a few common ones:

Example 3.1.1

Let $z_0, z_1 \in \mathbb{C}$. The line segment from z_0 to z_1 can be described by the curve $\gamma : [0,1] \to \mathbb{C}$ with:

$$\gamma(t) = tz_1 + (1-t)z_0 = z_0 + t(z_1 - z_0)$$

So, for example, the line segment from $1-i$ to $5+7i$ is given by the curve $\gamma(t) = 1-i+t(4+8i) = (1+4t)+(-1+8t)i$.

Example 3.1.2

Let $z_0 \in \mathbb{C}$ and $r \geq 0$. Fix an angle θ. Then travelling along the circle of radius r centered at z_0, starting at θ and travelling once counterclockwise, is done by the curve:

$$\gamma(t) = z_0 + re^{i(\theta+t)}$$

where $t \in [0, 2\pi]$.

If we travel clockwise instead, then the formula is $\gamma(t) = z_0 + re^{i(\theta-t)}$.

We will usually start at $\theta = 0$.

Also, notice that these are both closed curves.

There are a few interesting ways to manipulate and combine curves. These will pop up, so let's give a brief description:

Definition 3.1.2: $-\gamma$

Let $\gamma : [a,b] \to \mathbb{C}$ be a curve in $\mathbb{C}$. Then we can define the curve $-\gamma$ which traverses γ backwards at the same speed. We define $-\gamma : [a,b] \to \mathbb{C}$ by:

$$(-\gamma)(t) = \gamma(a+b-t)$$

Why $a+b-t$? When looking at γ, the variable t travels along the interval $[a,b]$ from a to b linearly. Going backwards, we want a linear function $s : [a,b] \to [a,b]$ so that $s(a) = b$ and $s(b) = a$. I.e., go from b to a linearly. The function $s(t) = a+b-t$ does this. It's linear, $s(a) = a+b-a = b$ and $s(b) = a+b-b = a$.

We can also combine curves, by following one and then the other.

Definition 3.1.3: $\gamma_1 + \gamma_2$

Suppose $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ are two curves such that $\gamma_1(b) = \gamma_2(c)$. I.e., the second curve starts where the first one ends.

Then $\gamma_1+\gamma_2$ is the curve that first follows γ_1 and then follows γ_2. We define $\gamma_1+\gamma_2 : [a, b+(d-c)] \to \mathbb{C}$ by:

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t), & t \in [a, b] \\ \gamma_2(c - b + t), & t \in [b, b + (d - c)] \end{cases}$$

The first part simply says to follow γ_1 until the end. The second part, however, looks confusing. Why $c - b + t$? Well, keep in mind that when we are done travelling γ_1, we are at $t = b$. To start γ_2, we need the input of γ_2 to be c. The formula $c - b + t$ gives $(\gamma_1 + \gamma_2)(b) = \gamma_2(c - b + b) = \gamma_2(c)$, like we want.

And then we end at $b + (d - c)$ since it gives $(\gamma_1 + \gamma_2)(b + (d - c)) = \gamma_2(c - b + (b + (d - c)) = \gamma_2(d)$, also as desired.

These are mostly academic formulas. They'll let us prove some things, and that's their only real use. In fact, we are very quickly going to prove a couple of theorems that render these formulas totally useless.

Lastly, we need to quickly talk a bit about closed curves. As it turns out, here's some notion of "direction" when you travel along a closed curve, provided that it doesn't self intersect.

Definition 3.1.4: Simple Closed Curve

A closed curve is called simple if it does not self-intersect. This means that $\gamma(t) = \gamma(s)$ occurs only when $t = s$ or at the start and end.

Theorem 3.1.1: Jordan Curve Theorem

Let γ be a simple closed curve in $\mathbb{C}$. Then γ divides $\mathbb{C}$ into two regions: the **inside** and **outside** of γ. We shall denote these $\text{in}(\gamma)$ and $\text{out}(\gamma)$.

The exterior of γ consists of all points $z \in \mathbb{C}$ such that there exists a path from z to ∞ that does not cross γ. The interior is $\mathbb{C} \setminus (\text{out}(\gamma) \cup \{\gamma(t)\})$.

Just to be clear, when we say that there is a path from z to ∞, we mean there is a continuous function $\sigma : [a, \infty) \to \mathbb{C}$ so that $\sigma(a) = z$ and $\lim_{t\to\infty} |\sigma(t)| = \infty$. I.e., the path gets arbitrarily far from the origin.

So, with that understanding, intuitively the outside is the set of all points that can "escape to ∞" without passing γ. The inside is all points that cannot escape to ∞ and which are not on γ.

The Jordan curve theorem is notoriously difficult to prove. For such a simple statement, you end up needing a lot of heavy machinery. (The first place I saw a proof was in graduate differential topology, to give you an idea of how hard it is to prove.) So we're naturally going to run far, far away.

However, this idea of inside and outside will let us define our notion of direction.

Definition 3.1.5: Orientation of a Simple Closed Curve

Let γ be a simple, closed curve. Then γ is **positively oriented** if the inside of γ is on your left as you traverse γ. If the inside of γ remains on your right, the curve is **negatively oriented**.

This is a visual definition. Orientation is something you need to be able to understand from a picture. There is a more formal definition, but it isn't terribly useful for us.

Example 3.1.3

The curve which follows the triangle from 0 to 1 to i and back to 0 is positively oriented.

Circles travelled counterclockwise are positively oriented. Circles travelled clockwise are negatively oriented.

Example 3.1.4

If γ is positively oriented, then $-\gamma$ is negatively oriented. If you travel the curve backwards, the inside of γ now appears on the opposite side.

3.2 Line Integrals

Now that we've talked a whole bunch about curves, let's talk about how to compute a complex line integral.

Definition 3.2.1: Line integrals in $\mathbb{C}$

Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve in $\mathbb{C}$ and $f = u + iv$ be a complex function whose domain contains γ. Let $\gamma(t) = a(t) + ib(t)$. Then we define the line integral of f over γ by:

$$\begin{aligned}\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \\ &= \int_a^b (u(\gamma(t)) + iv(\gamma(t)))(a'(t) + ib'(t))dt \\ &= \int_a^b u(a(t), b(t))a'(t) - v(a(t), b(t))b'(t)dt + i\int_a^b v(a(t), b(t))a'(t) + u(a(t), b(t))b'(t)dt\end{aligned}$$

Before we develop all sorts of neat techniques, let's get our hands dirty and actually compute a line integral.

Example 3.2.1

Let γ be the line segment from $2 - i$ to $-3 + 2i$. Find $\int_\gamma e^z dz$.

We have $\gamma(t) = (2 - i) + t((-3 + 2i) - (2 - i)) = (2 - i) + t(-5 + 3i) = (2 - 5t) + i(-1 + 3t)$, where

t goes from 0 to 1. As such, $\gamma'(t) = -5 + 3i$.

$$\begin{aligned}\int_\gamma e^z dz &= \int_0^1 e^{(2-5t)+i(-1+3t)}(-5+3i)dt \\ &= \int_0^1 (e^{2-5t}\cos(3t-1) + ie^{2-5t}\sin(3t-1))(-5+3i)dt \\ &= \int_0^1 -5e^{2-5t}\cos(3t-1) - 3e^{2-5t}\sin(3t-1)dt + i\int_0^1 3e^{2-5t}\cos(3t-1) - 5e^{2-5t}\sin(3t-1)dt\end{aligned}$$

And this has now become a particularly nasty first year calculus integral. Integrating by parts a couple of times gives an answer. But there must be a better way?

Rather than continuing to bash our heads against this, let's try to find a smarter way to handle this integral. After all, we would expect integrating a simple function like e^z to be simple. If we were to integrate $\int_a^b e^x dx$, we know by the Fundamental Theorem of Calculus that this is $e^b - e^a$. Is there a complex version of this?

Theorem 3.2.1: The Complex Fundamental Theorem of Calculus

Suppose $F(z)$ is an analytic function on an open set U, and $\gamma : [a, b] \to \mathbb{C}$ is a smooth curve contained in U. If $F'(z) = f(z)$, then:

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

Proof. The key tool here is the chain rule: $F'(\gamma(t))\gamma'(t) = (F \circ \gamma)'(t)$. This is a slightly different chain rule that we've seen so far, so let's give a quick proof:

Let $F(z) = u(x, y) + iv(x, y)$ and $\gamma(t) = a(t) + ib(t)$. Then $F(\gamma(t)) = u(a(t), b(t)) + iv(a(t), b(t))$. Now, $F(\gamma(t))$ is a curve in $\mathbb{C}$, so we know that its derivative is:

$$(F \circ \gamma)(t) = \frac{d\,u(a(t), b(t))}{d\,t} + i\frac{d\,v(a(t), b(t))}{d\,t}$$

Now, by the chain rule for functions from $\mathbb{R}$ to $\mathbb{R}^2$, we have:

$$\frac{d\,u(a(t), b(t))}{d\,t} = \frac{\partial u(a(t), b(t))}{\partial x}a'(t) + \frac{\partial u(a(t), b(t))}{\partial y}b'(t)$$

With this, we now compute the derivative of $F \circ \gamma$:

$$(F \circ \gamma)'(t) = u_x(a(t), b(t))a'(t) + u_y(a(t), b(t))b'(t) + i\,(v_x(a(t), b(t))a'(t) + v_y(a(t), b(t))b'(t))$$

On the other hand, we know that $F' = u_x + iv_x$, and that $u_x = v_y$ and $u_y = -v_x$ by the Cauchy-Riemann

equations. As such:

$$\begin{aligned}
F'(\gamma(t))\gamma'(t) &= (u_x(a(t),b(t)) + iv_x(a(t),b(t)))(a'(t)+ib'(t)) \\
&= u_x(a(t),b(t))a'(t) - v_x(a(t),b(t))b'(t) + i\left(v_x(a(t),b(t))a'(t) + u_x(a(t),b(t))b'(t)\right) \\
&= u_x(a(t),b(t))a'(t) + u_y(a(t),b(t))b'(t) + i\left(v_x(a(t),b(t))a'(t) + v_y(a(t),b(t))b'(t)\right) \\
&= (F\circ\gamma)'(t)
\end{aligned}$$

With this intermediate result in hand, we can now tackle what we wish to prove. We have:

$$\int_\gamma f(z)dz = \int_a^b F'(\gamma(t))\gamma'(t)dt = \int_a^b (F\circ\gamma)'(t)dt$$

Let $(F\circ\gamma)(t) = x(t)+iy(t)$. So then $(F\circ\gamma)'(t) = x'(t)+iy'(t)$. As such:

$$\int_\gamma f(z)dz = \int_a^b x'(t)dt + i\int_a^b y'(t)dt$$

However, since x and y are real functions, we can now use the Fundamental Theorem of Calculus over $\mathbb{R}$ to get:

$$\begin{aligned}
\int_\gamma f(z)dz = x(b)-x(a)+i(y(b)-y(a)) \\
&= (x+iy)(b)-(x+iy)(a) \\
&= (F\circ\gamma)(b)-(F\circ\gamma)(a)
\end{aligned}$$

as desired. □

Example 3.2.2

Find $\int_\gamma e^z dz$ where γ is the line segment from $2-i$ to $-3+2i$.

Well, we know that if $f(z)=e^z$, then $f'(z)=e^z=f(z)$. So, we have that:

$$\int_\gamma e^z dz = e^{\text{end of }\gamma} - e^{\text{start of }\gamma} = e^{-3+2i} - e^{2-i}$$

Unlike over $\mathbb{R}$, we don't call a function $F(z)$ so that $F'(z)=f(z)$ an antiderivative. The terminology in $\mathbb{C}$ is a primitive.

Definition 3.2.2: Primitive

Let $f(z)$ be a function defined on an open set $U\subset\mathbb{C}$. We say that an analytic function $F(z)$ is a **primitive** for $f(z)$ on U if $F'(z)=f(z)$ for all $z\in U$.

At this point, we could reduce this to a course in finding primitives. However, as you've seen in first year, that's a wholly unsatisfying endeavour. Also, it ignores the uniquely complex characteristics from our setting.

3.2.1 Green's Theorem and the Cauchy Integral Theorem

As a quick refresher from multivariable calculus, let's remind ourselves what Green's theorem says.

Theorem 3.2.2: Green's Theorem

Let γ be a positively oriented, piecewise smooth, simple, closed curve in $\mathbb{R}^2$. Suppose that $f, g : U \to \mathbb{R}$ where U is an open set containing both γ and the inside $\text{in}(\gamma)$. If f and g have continuous partials, then:

$$\int_\gamma f dx + g dy = \iint_{\text{in}(\gamma)} (g_x - f_y) dx dy$$

We will not be proving this. We will, however, gladly use it. Before we use it, let's unpack the line integral. Let $\gamma : [a, b] \to \mathbb{R}^2$ be given by $\gamma(t) = (x(t), y(t))$. Then:

$$\int_\gamma f dx + g dy = \int_a^b f(x(t), y(t))x'(t) + g(x(t), y(t))y'(t) dt$$

We can use this to prove a very useful theorem in $\mathbb{C}$. However, it is going to require us to make an assumption. We are going to assume for the remainder of the course that if $f(z)$ is holomorphic on an open set U, then u_x, u_y, v_x, and v_y are all continuous on U. (It is my intent to revisit this later and provide a proof, as an appendix.)

With this assumption, we can prove:

Theorem 3.2.3: Cauchy's Integral Theorem (Version 1)

Suppose f is holomorphic on an open set U, and γ is a piecewise smooth, positively oriented, simple, closed curve in U, such that $\text{in}(\gamma) \subset U$ as well. Then:

$$\int_\gamma f(z) dz = 0$$

Proof. This turns out to be a quick application of Green's theorem. We have:

$$\int_\gamma f(z) dz = \int_a^b u(a(t), b(t))a'(t) - v(a(t), b(t))b'(t) dt + i \int_a^b v(a(t), b(t))a'(t) + u(a(t), b(t))b'(t) dt$$

Notice that $\int_a^b u(a(t), b(t))a'(t) - v(a(t), b(t))b'(t) dt = \int_\gamma u dx - v dy$. So by Green's theorem, this becomes:

$$\int_a^b u(a(t), b(t))a'(t) - v(a(t), b(t))b'(t) dt = \int_\gamma u dx - v dy = \iint_{\text{in}(\gamma)} (-v_x) - u_y dx dy$$

However, since f is holomorphic, we have that $-v_x = u_y$. So:

$$\int_a^b u(a(t), b(t))a'(t) - v(a(t), b(t))b'(t) dt = \iint_{\text{in}(\gamma)} u_y - u_y dx dy = 0$$

Similarly, we also find that:

$$\int_a^b v(a(t), b(t))a'(t) + u(a(t), b(t))b'(t)dt = \iint_{\text{in}(\gamma)} u_x - v_y dxdy = \iint_{\text{in}(\gamma)} u_x - u_x dxdy = 0$$

As such, $\int_\gamma f(z)dz = 0 + i0 = 0$. □

Why go through all this effort to prove this? Shouldn't this just be a quick application of the ℂFTC?

Example 3.2.3

Consider this argument:

Proof. Let F be a primitive for f on the open set U. Then by ℂFTC:

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

However, we know that γ is closed, so $\gamma(a) = \gamma(b)$. As such:

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(b)) = 0$$

□

What's wrong with this proof? Why did we have to break out Green's theorem instead?

Well, this is predicated on us having a primitive F for f! We don't know that analytic functions have primitives on any open set, and it turns out not to be true.

So when does an analytic function have a primitive? How might we go about finding this function? For inspiration, we turn to a similar result from first year calculus, the Fundamental Theorem of Calculus.

Recall that the FTC has two parts. The first part we already have an analogue for, which is our ℂFTC. The second part says that if $f(x)$ is a continuous function on $[a, b]$, then:

$$F(x) = \int_a^x f(t)dt$$

is a differentiable function on (a, b) with $F'(x) = f(x)$. I.e., that F is an antiderivative for f.

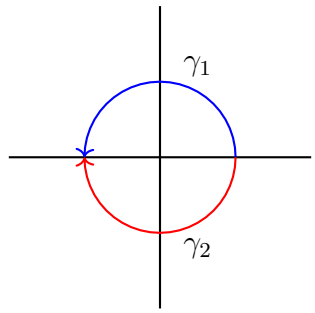
If we try to emulate this definition in ℂ, we could try to define a function $F(z)$ on a domain D as follows. Fix a point z_0 in D. For any point $z \in D$, we define

$$F(z) = \int_{\gamma_z} f(w)dw$$

where γ_z is any curve from z_0 to z in D. Does this definition make sense? Well, we can compute these integrals for sure. So let's work out an example.

Example 3.2.4

Let $D = \mathbb{C} \setminus \{0\}$, and $f(z) = \frac{1}{z}$. This function is analytic on the domain D. We'll fix our point $z_0 = 1$. Let's consider what this formula gives $F(-1)$. We will compute the integral along two curves:



The curve γ_1 is the upper semicircle of radius 1 centered at 0, travelled from 1 to -1. We can parametrize it as $\gamma_1(t) = e^{it}$ for $t \in [0, \pi]$. Its derivative is $\gamma_1'(t) = ie^{it}$. So we compute the integral:

$$\int_{\gamma_1} \frac{1}{z} dz = \int_0^\pi \frac{1}{e^{it}} ie^{it} dt = \int_0^\pi i dt = i\pi$$

The curve γ_2 is the lower semicircle of radius 1 centered at 0, travelled from 1 to -1. We can parametrize it as $\gamma_2(t) = e^{-it}$ for $t \in [0, \pi]$. Its derivative is $\gamma_2'(t) = -ie^{it}$. So we compute the integral:

$$\int_{\gamma_2} \frac{1}{z} dz = \int_0^\pi -\frac{1}{e^{-it}} ie^{-it} dt = \int_0^\pi -i dt = -i\pi$$

We find that $-i\pi = F(-1) = i\pi$, which tells us that F isn't a function!

In this example, we ran afoul of something very unfortunate: complex line integration is not path independent in general.

3.2.2 Path Independence

Let's investigate path independence of integrals. When can we guarantee that $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$? To begin, let's investiage this equation a bit more. If these integrals are equal, then we can rearrange to get:

$$\begin{aligned}
0 &= \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz \\
&= \int_{\gamma_1} f(z)dz + \int_{-\gamma_2} f(z)dz && \text{From week 5 homework, Q6} \\
&= \int_{\gamma_1 + (-\gamma_2)} f(z)dz && \text{From week 5 homework, Q7}
\end{aligned}$$

So, the integrals $\int_{\gamma_1} f(z)dz$ and $\int_{\gamma_2} f(z)dz$ are equal if and only if the integral $\int_{\gamma_1 - \gamma_2} f(z)dz$, over the closed

curve $\gamma_1 + (-\gamma_2)$ is 0. This looks an awful lot like a Cauchy Integral Theorem type result. However, we have no guarantees that $\gamma_1 + (-\gamma_2)$ is smooth or simple. We only know that it is piecewise smooth and closed.

So, for the moment, let's try to generalize the Cauchy-Integral theorem to handle piecewise smooth, closed curves. To do so, we need to overcome several hurdles. We need to:

- define integration over piecewise smooth curves
- show we can generalize CIT to handle piecewise smooth curves
- show we can generalize CIT to handle non-simple closed curves

So, to begin, we define:

Definition 3.2.3

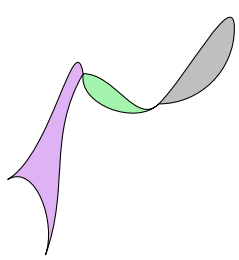
Let γ be a piecewise smooth curve. Then γ can be written as $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$, where each γ_j is a smooth curve.

We define the integral $\int_\gamma f(z)dz$ as:

$$\int_\gamma f(z)dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z)dz$$

Now, fortunately, extending CIT to work over piecewise smooth curves takes no work. Green's theorem applies to piecewise smooth, simple, closed curves as well.

To handle the closed case, the idea is fairly simple. If we have a closed curve, such as:



which is composed of three different piecewise smooth, closed curves. If f is analytic on a domain containing each of these curves and their insides, then we can use CIT on each of them. Then the total integral will be the sum of each of these integrals, which will be 0.

A complete proof of this is much more technical. So our proof will be somewhat "hand-wavey".

Theorem 3.2.4: Cauchy's Integral Theorem (version 2)

Let f be a function that is analytic on a domain D. Suppose that γ is a closed, piecewise smooth curve

such that D contains γ and all of the regions bounded by γ. Then:

$$\int_\gamma f(z)dz = 0$$

Proof. The conditions on γ ensure that γ can be decomposed into:

- countably many piecewise smooth, simple closed curves
- countably many curves σ_j such that γ traverses each σ_j an equal number in both directions

As such, we have that:

$$\int_\gamma f(z)dz = \sum_{\gamma_j \text{ is a simple, closed summand of} \gamma} \int_{\gamma_j} f(z)dz + \sum_{\sigma_j \text{ is a curve such that } \gamma \text{ traverses } \sigma_j \text{ an equal number of times in each direction}} n\left(\int_{\sigma_j} f(z)dz + \int_{-\sigma_j} f(z)dz\right)$$

By our original CIT, the first summand is 0. And since $\int_{\sigma_j} f(z)dz + \int_{-\sigma_j} f(z)dz = 0$, the second summand is 0. Therefore:

$$\int_\gamma f(z)dz = 0$$

□

We can use this to give one condition on path independence:

Theorem 3.2.5

Suppose $f(z)$ is analytic on a domain D containing the two curves γ_1, γ_2, which start and end at the same point, and which contains all points bounded between γ_1, γ_2. Then:

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

Proof. By CIT version 2:

$$\int_{\gamma_1 - \gamma_2} f(z)dz = 0$$

We showed earlier that this is equivalent to $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$. □

The conditions on the curves in our CIT version 2 and in theorem 3.2.5 – namely that the domain contains all regions bounded by the curve – are generally annoying to handle on a case by case basis. And if we want

our strategy to find a primitive to work, we need these conditions to hold for all curves from z_0 to z in D. So we should expect this to require a condition on D. Fortunately, there's a nice topological condition we can impose that guarantees everything we need.

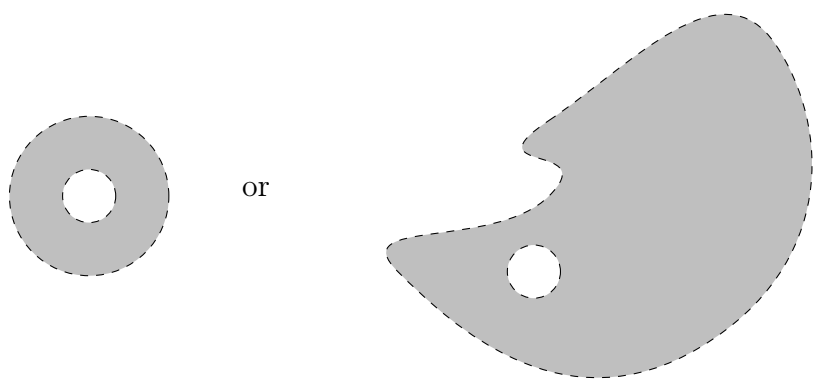
Definition 3.2.4: Simply Connected Domains

A domain D is called simply connected if for every simple, closed curve γ in D, that $\text{in}(\gamma) \subset D$.

Intuitively, this means that the set has no "holes". For example:

Example 3.2.5

These domains are not simply connected because they have "holes":



On the other hand, there are plenty of sets we're used to that are simply connected. $\mathbb{C}$, $\mathbb{C} \setminus (-\infty, 0]$, and any open ball are all simply connected.

This condition lets us state another version of the Cauchy Integral Theorem:

Theorem 3.2.6: Cauchy's Integral Theorem (version 3)

Suppose $f(z)$ is analytic on a simply connected domain D. If γ is any piecewise smooth, closed curve in D, then:

$$\int_\gamma f(z)dz = 0$$

Proof. This follows immediately from CIT version 2, since any closed curve in a simply connected domain satisfies the hypotheses of CIT version 2. □

Having a sufficiently general version of the Cauchy Integral Theorem is a key step towards proving results about primitives. However, before we do that, we need another result. As it turns out, the result guaranteeing primitives that I would like to prove hinges on the ability to estimate integrals. So, before we can finish talking about primitives, we need the following result:

Theorem 3.2.7: M-L Estimation of Integrals

Suppose $\gamma : [a, b] \to \mathbb{C}$ is a piecewise smooth curve and $f(z)$ is a continuous function whose domain includes γ. Let $M = \max\{|f(\gamma(t))| | t \in [a, b]\}$, and $L = \text{Length}(\gamma)$. Then:

$$\left| \int_\gamma f(z)dz \right| \leq ML$$

Proof. To begin, we will need to prove a nice fact: for any curve $g : [a, b] \to \mathbb{C}$, we have:

$$\left| \int_a^b g(t)dt \right| \leq \int_a^b |g(t)|dt$$

Let $\int_a^b g(t)dt = re^{i\theta}$. Set $h(t) = e^{-i\theta}g(t)$. Then:

$$\int_a^b h(t)dt = \int_a^b e^{-i\theta}g(t)dt = e^{-i\theta}\int_a^b g(t)dt = r$$

So $\int_a^b h(t)dt \in \mathbb{R}$. We therefore have that $\text{Re}\int_a^b h(t)dt = \int_a^b \text{Re}(h(t))dt$. So, we have:

$$\begin{aligned}
\left| \int_a^b g(t)dt \right| &= r \\
&= \left| \int_a^b h(t)dt \right| \\
&= \left| \int_a^b \text{Re}(h(t))dt \right| \\
&\leq \int_a^b |\text{Re}(h(t))|\, dt && \text{(from multivariable calc)} \\
&\leq \int_a^b |h(t)|dt && \text{(since } |\text{Re}(h(t))| \leq |h(t)|\text{)} \\
&= \int_a^b |g(t)|dt && \text{(since } |h(t)| = |e^{-i\theta}||g(t)| = |g(t)|\text{)}
\end{aligned}$$

With this in hand, we can now proceed to prove the result we desire:

$$\begin{aligned}
\left| \int_\gamma f(z)dz \right| &= \left| \int_a^b f(\gamma(t))\gamma'(t)dt \right| \\
&\leq \int_a^b |f(\gamma(t))||\gamma'(t)|dt \\
&\leq \int_a^b M|\gamma'(t)|dt \\
&= M\int_a^b |\gamma'(t)|dt
\end{aligned}$$

However, we know that $\int_a^b |\gamma'(t)|dt = L$, and so we have the desired inequality. $\square$

This result is, for now, theoretically interesting. However, much later on in the course, it will become very useful in practice.

Now that we have a better Cauchy's Integral Theorem and this technical result, we can prove:

Theorem 3.2.8

If $f(z)$ is analytic on a simply connected domain D, then f has a primitive on this domain.

Proof. Fix a point z_0 in D. For any point $z \in D$, we define

$$F(z) = \int_{\gamma_z} f(w)dw$$

where γ_z is any piecewise smooth curve from z_0 to z. By CIT version 3, we know that integration of f in D is path independent, so this function is well defined.

All that remains is to show that $F'(z) = f(z)$. Let γ_z be any curve from z_0 to z. Since D is a domain, there exists a radius $r > 0$ such that $B_r(z) \subset D$, so we assume that $|h| < r$. Let γ_h be the straight line from z to h. Notice that $\gamma_h \subset B_r(z)$, and that $\gamma_z + \gamma_h$ is a curve from z_0 to $z + h$ in D. As such:

$$F'(z) = \lim_{h\to 0} \frac{\int_{\gamma_{z+h}} f(w)dw - \int_{\gamma_z} f(w)dw}{h} = \lim_{h\to 0} \frac{1}{h} \int_{\gamma_h} f(w)dw$$

Using this, we find:

$$\begin{aligned}
|F'(z) - f(z)| &= \left| \lim_{h\to 0} \left(\frac{1}{h} \int_{\gamma_h} f(w)dw \right) - f(z) \right| \\
&= \left| \lim_{h\to 0} \frac{1}{h} \int_{\gamma_h} f(w)dw - \frac{1}{h} \int_{\gamma_h} f(z)dw \right| \\
&= \left| \lim_{h\to 0} \frac{1}{h} \int_{\gamma_h} (f(w) - f(z))dw \right|
\end{aligned}$$

Now, since f is continuous on D, then for any $\varepsilon > 0$ there exists $r > 0$ such that if $|w| < r$, then $|f(w) - f(z)| < \varepsilon$. If $|h| < r$, then for any w on γ_h, we also have $|w| < r$. This gives us that $M = \max\{f(w)|w = \gamma_h(t)\} < \varepsilon$. Notice also that γ_h has length $|h|$. So by M-L estimation, we have:

$$|F'(z) - f(z)| \leq \left| \lim_{h\to 0} \frac{1}{h} \varepsilon |h| \right| = \lim_{h\to 0} \frac{|h|}{|h|} \varepsilon = \varepsilon$$

Therefore, for any $\varepsilon > 0$, $|F'(z) - f(z)| < \varepsilon$. So $|F'(z) - f(z)| = 0$ and $F'(z) = f(z)$. $\square$

Alright, so this is fairly technical. However, it tells us that a lot of functions have primitives. However, note that this is only one direction. This does not say that if the domain isn't simply connected, then f has no primitive.

Example 3.2.6

We know that $\frac{d\frac{1}{z}}{dz} = -\frac{1}{z^2}$. So, even though $\mathbb{C} \setminus \{0\}$ is not simply connected, $\frac{1}{z^2}$ has a primitive!

On the other hand, this does tell us that some fairly fantastical functions have primitives. For example, $\sin(\sin(\sin(e^{z\cos(z)}) + z^2))\cos(z^3 - 1)$ has a primitive on $\mathbb{C}$! Good luck finding it, but it's there.

Alright, so we have a result that tells us when a function has a primitive. Can we figure out when a function doesn't have a primitive? Well, remember that if f has a primitive on D, then the integral of f over any closed curve is automatically 0, by $\mathbb{C}$FTC. This turns out to actually be an if and only if:

Theorem 3.2.9

Let f be analytic on a domain D. Then f has a primitive on D if and only if $\int_\gamma f(z)dz = 0$ for any piecewise smooth, closed curve in D.

Proof. If f has a primitive, $\mathbb{C}$FTC gives that $\int_\gamma f(z)dz = 0$.

On the other hand, if $\int_\gamma f(z)dz = 0$ for *every* γ, then our argument in simply connected case still applies. The integral definition of $F(z)$ is still well-defined. The rest of the argument doesn't use that D is simply connected, and so the rest of the proof still applies! $\square$

In practice, we can use this to conclude when something doesn't have a primitive on a domain.

Example 3.2.7

$f(z) = \frac{1}{z}$ does not have a primitive on any domain containing the unit circle. To prove this, note that we have shown that the integral of $f(z)$ from 1 to -1 over the upper unit circle is $i\pi$, while over the lower unit circle we have $-i\pi$. All together, this tells us that the integral of $f(z)$ over the whole unit circle is $2\pi i$.

Since there is a closed curve γ in D with $\int_\gamma f(z)dz \neq 0$, we cannot have a primitive.

3.3 A Roadmap

So far, we have seen how to handle a large class of integrals. integrating analytic functions on simply connected domains is easy. How do we handle the case where we want to integrate over a domain which is not simply connected? For example, how do we find:

$$\int_{|z|=2} \frac{1}{z^4+1} dz \qquad \text{or} \qquad \int_{|z|=1} e^{\frac{1}{z}} dz$$

The first of these functions is analytic on $\mathbb{C} \setminus \{\pm\frac{1+i}{\sqrt{2}}, \pm\frac{1-i}{\sqrt{2}}\}$, and the second is analytic on $\mathbb{C} \setminus \{0\}$. Neither of their domains contains the inside of the circles. So we cannot apply the Cauchy Integral Theorem.

Is there a unifying theory that will tells us how to handle integrals like this? It turns out there is. It hinges around the notion of an "isolated singularity".

Definition 3.3.1

An **isolated singularity** $z_0 \in \mathbb{C}$ of a function f is a point such that:

- f is discontinuous at z_0
- f is analytic on $B_r(z_0) \setminus \{z_0\}$ for some $r > 0$

So, an isolated singularity is a point where the function is not continuous, but is analytic around it.

Example 3.3.1

$\frac{1}{z}$ has an isolated singularity at $z = 0$.

$\frac{1}{z^4-1}$ has isolated singularities at $z = \pm 1, \pm i$.

$\text{Log}(z)$ has no isolated singularities.

The strategy for integrating on curves whose inside contains isolated singularity depends on the type of singularity. There are three types: removable discontinuities, poles, and essential singularities.

Example 3.3.2

$\frac{z^2-1}{z-1}$ has a removable discontinuity at $z = 1$. $\frac{\sin(z)}{z}$ has a removable discontinuity at $z = 0$.

Definition 3.3.2: Removable Discontinuity

A **removable discontinuity** z_0 is an isolated singularity such that:

$$\lim_{z \to z_0} f(z) \text{ exists}$$

So, a removable discontinuity is a an isolated singularity which can be filled. It turns out that filling this discontinuity results in an analytic function, and so the Cauchy Integral Theorem still applies in this case!

Definition 3.3.3: Pole of order n

A **pole of order** n is an isolated singularity z_0 such that there exists a function $g(z)$ which is analytic on $B_r(z_0)$, $g(z_0) \neq 0$, and:

$$f(z) = \frac{g(z)}{(z - z_0)^n}$$

A pole of order 1 is called a **simple pole**.

These isolated singularities behave like vertical asymptotes. We will see later on that $\lim_{z \to z_0} f(z) = \infty$, and the order tells you how quickly the function tends to ∞.

Example 3.3.3

$\frac{1}{z^4-1}$ has simple poles at each of $\pm 1, \pm i$. For example, at $z = 1$, we can write $g(z) = \frac{1}{(z+1)(z^2+1)}$ and $\frac{1}{z^4-1} = \frac{g(z)}{z-1}$. The function $g(z)$ is analytic on $\mathbb{C} \setminus \{-1, i, -i\}$, and so in particular it is analytic on $B_1(1)$.

Lastly, we have the worst behaved of the lot.

Definition 3.3.4: Essential Singularity

An **essential singularity** is an isolated singularity that is neither a pole nor removable.

It turns out that this type of singularity behaves terribly. Not only does $\lim_{z \to z_0} f(z)$ not exist, it's also not ∞. Furthermore, it turns out that for any $r > 0$, $\{f(w) | w \in B_r(z_0) \setminus \{z_0\}\} = \mathbb{C}$ or $\mathbb{C} \setminus \{0\}$. This result, which we won't prove, is called the Great Picard's Theorem. So, $f(z)$ takes on every value (except maybe one value) infinitely often near its essential singularities. This is incredibly bad behavior. This is akin to a function like $\frac{\sin\left(\frac{1}{x}\right)}{x}$ on $\mathbb{R}$.

For each of these types of isolated singularity, we're going to have a particular method:

- To integrate around removable discontinuities, we will soon see that the Cauchy Integral Theorem is sufficient.
- To integrate around a pole, we will need to talk about the Cauchy Integral Formula.
- To integrate around essential singularities, we are going to need to talk about power series and Laurent series. This will lead us to a nice result, called the Residue theorem, which will encapsulate each of these methods.

3.4 Integrating Around Poles

Our first stop along this roadmap are functions with poles. How do we integrate $\int_\gamma f(z)dz$ if $f(z)$ has poles inside γ? In particular, we're going to discuss simple poles first. For example, how do we find:

$$\int_{|z|=1} \frac{e^z}{z} dz$$

Since this function isn't analytic inside the curve, the Cauchy Integral Theorem does not apply. We have no reasonable guess as to a primitive (and indeed, this function does not have a primitive on any domain containing the unit circle!). What if we try using the definition of the integral?

Example 3.4.1

Let $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. Then:

$$\int_\gamma \frac{e^z}{z} dz = \int_0^{2\pi} \frac{e^{e^{it}}}{e^{it}} ie^{it} dt = i \int_0^{2\pi} e^{\cos(t)}(\cos(\sin(t)) + i\sin(\sin(t)))dt$$

How exactly are we supposed to evaluate this mess of an integral? It doesn't appear likely that any of the techniques from first year calculus will be of much use here.

So, none of our techniques so far work. We need something new. It turns out there is a general theorem that will let us handle any function with a simple pole.

3.4.1 Cauchy's Integral Formula

The key result for integrating around poles is called Cauchy's integral formula.

Theorem 3.4.1: Cauchy's Integral Formula

Suppose $f(z)$ is analytic on a domain D such that $\{z \in \mathbb{C} | |z - z_0| \leq r\} \subset D$. Let γ be the circle of radius r centered at z_0, travelled once counterclockwise. Then:

$$\int_\gamma \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

Before we prove this, let's see how it applies to our previous example.

Example 3.4.2

Let $f(z) = e^z$. This is entire, so is analytic on $\mathbb{C}$. $\mathbb{C}$ is a domain containing $\{z \in \mathbb{C} | |z - z_0| \leq 1\}$. Let $|z| = 1$ refer to the unit circle travelled once counterclockwise. Then CIF applies to give:

$$\int_{|z|=1} \frac{e^z}{z} dz = \int_{|z|=1} \frac{f(z)}{z - 0} dz = 2\pi i f(0) = 2\pi i e^0 = 2\pi i$$

Notice: the theorem is very easy to apply. No complicated arithmetic involved. However, we do need to check the conditions of the theorem before we apply it. This takes some care.

Let's prove this theorem. The proof contains at least one very useful idea.

Proof. Let γ_s be the circle of radius s centered at z_0, where $s \in (0, r]$. We may assume, without loss of generality, that each γ_s starts at the angle $\theta = 0$ and ends at $\theta = 2\pi$. We define a function $F(s)$ as follows:

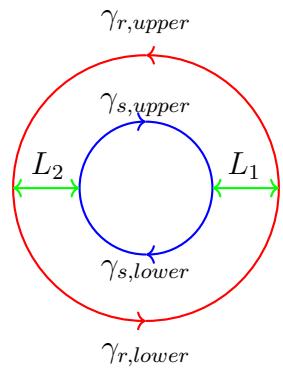
$$F(s) = \int_{\gamma_s} \frac{f(z)}{z - z_0} dz$$

So the integral we are interested in is $F(r)$. We shall prove two facts about $F(s)$:

- $F(s)$ is constant on $(0, r]$.

- $\lim_{s\to 0^+} F(s) = 2\pi i f(z_0)$.

To prove that $F(s)$ is constant, consider the following picture:



where L_1 and L_2 travel from left to right.

Notice that $-\gamma_s = \gamma_{s,upper} + \gamma_{s,lower}$ (since $\gamma_{s,upper}$ and $\gamma_{s,lower}$ are travelling clockwise) and that $\gamma_r = \gamma_{r,upper} + \gamma_{r,lower}$.

Now, we know that $\frac{f(z)}{z-z_0}$ is analytic on $D \setminus \{z_0\}$, which is a domain containing the closed curve $\gamma_{r,upper} + L_2 + \gamma_{s,upper} + L_1$. So by CIT:

$$\int_{\gamma_{r,upper}+L_2+\gamma_{s,upper}+L_1} \frac{f(z)}{z-z_0} dz = 0$$

And similarly:

$$\int_{\gamma_{r,lower}-L_1+\gamma_{s,lower}-L_2} \frac{f(z)}{z-z_0} dz = 0$$

Adding these two together gives us that:

$$\int_{\gamma_{r,upper}} \frac{f(z)}{z-z_0} dz + \int_{L_2} \frac{f(z)}{z-z_0} dz + \int_{\gamma_{r,lower}} \frac{f(z)}{z-z_0} dz + \int_{L_1} \frac{f(z)}{z-z_0} dz$$
$$+ \int_{\gamma_{s,upper}} \frac{f(z)}{z-z_0} dz + \int_{-L_1} \frac{f(z)}{z-z_0} dz + \int_{\gamma_{s,lower}} \frac{f(z)}{z-z_0} dz + \int_{-L_2} \frac{f(z)}{z-z_0} dz = 0$$

After simplifying, we find that:

$$\int_{\gamma_r} \frac{f(z)}{z-z_0} dz + \int_{-\gamma_s} \frac{f(z)}{z-z_0} dz = 0$$

As such, $F(s) = F(r)$ for all $s \in (0, r)$. So F is constant on $(0, r]$.

So, we see that $\lim_{s\to 0^+} F(s) = \lim_{s\to 0^+} F(r) = F(r)$.

All that remains is for us to actually compute this limit. To do that, we go to the definition of the integral.

$$\int_{\gamma_s} \frac{f(z)}{z-z_0}dz = \int_0^{2\pi} \frac{f(z_0+se^{it})}{se^{it}} ise^{it}dt = \int_0^{2\pi} if(z_0+se^{it})dt$$

We claim that $\lim s \to 0^+ \int_0^{2\pi} if(z_0+se^{it})dt = 2\pi i f(z_0)$. Consider the difference:

$$\begin{aligned}
\lim s \to 0^+ \left| \int_0^{2\pi} if(z_0+se^{it})dt - 2\pi i f(z_0) \right| &= \lim_{s\to 0^+} \left| \int_0^{2\pi} if(z_0+se^{it})dt - \int_0^{2\pi} if(z_0)dt \right| \\
&\leq \lim_{s\to 0^+} \int_0^{2\pi} |f(z_0+se^{it}) - f(z_0)|dt \\
&\leq \lim_{s\to 0^+} 2\pi \max\{|f(w)-f(z_0)| | w \in \{z \in \mathbb{C} | |z-z_0| \leq s\}\}
\end{aligned}$$

Now, since f is continuous at z_0, we see that $\lim_{s\to 0^+} \max\{|f(w)-f(z_0)| | w \in \{z \in \mathbb{C} | |z-z_0| \leq s\}\} = 0$. So the squeeze theorem gives us that:

$$\lim_{s\to 0^+} \left| \int_0^{2\pi} if(z_0+se^{it})dt - 2\pi i f(z_0) \right| = 0$$

As such, $\lim_{s\to 0^+} \int_0^{2\pi} if(z_0+se^{it})dt = 2\pi i f(z_0)$ as desired.

Putting it all together, we get that:

$$\int_{\gamma_r} \frac{f(z)}{z-z_0}dz = F(r) = \lim_{s\to 0^+} F(s) = 2\pi i f(z_0)$$

□

While technical, this proof has a really important idea. The technique for showing that the integrals over the two circles are equal is fairly useful. For example:

Example 3.4.3

Let γ be the circle of radius 3 centered at 0, travelled once counterclockwise. Find $\int_\gamma \frac{1}{z-1}dz$.

As written, CIF doesn't apply. This first version of CIF only applies to circles centered at z_0, which in this case is 1.

However, our technique from the proof of CIF gives that:

$$\int_\gamma \frac{1}{z-1}dz = \int_{|z-1|=1} \frac{1}{z-1}dz$$

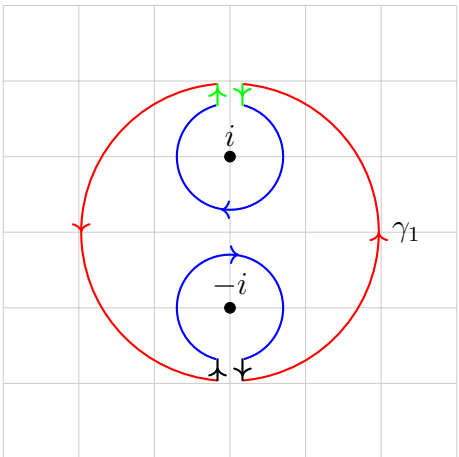
And now we're in a situation where CIF applies. It gives $\int_\gamma \frac{1}{z-1}dz = 2\pi i$.

We continue with another example of how to use the Cauchy Integral Formula:

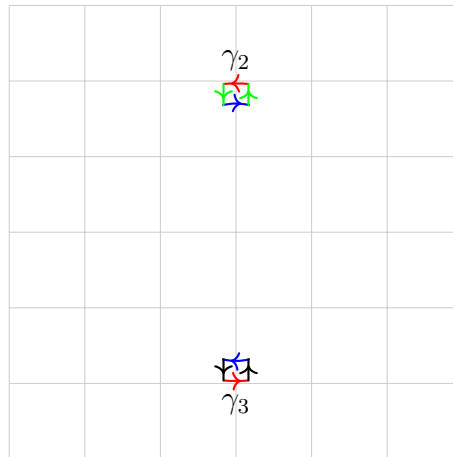
Example 3.4.4

Let γ be the circle of radius 2 centered at 0, travelled once counterclockwise. Find $\int_\gamma \frac{1}{z^2+1}dz$.

The function $\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$ has two simple poles inside the curve. So CIF doesn't immediately apply. Instead, we need to turn this into a situation where it does. Consider the following curves:



and



Since none of these three curves enclose $\pm i$, CIT applies to give that:

$$\int_{\gamma_j} \frac{1}{z^2+1}dz = 0$$

for each $j = 1, 2, 3$. Adding them together gives that:

$$0 = \sum_{j=1}^{3} \int_{\gamma_j} \frac{1}{z^2+1}dz = \int_C \frac{1}{z^2+1}dz - \int_{C_1} \frac{1}{z^2+1}dz - \int_{C_2} \frac{1}{z^2+1}dz$$

where C is the large circle travelled once counterclockwise, C_1 is the smaller circle around i travelled once counterclockwise, and C_2 is the smaller circle around $-i$ travelled once counterclockwise. This second equality comes from noticing that the green lines in γ_1 and γ_2 are travelled in opposite directions, so their integrals cancel out. The same is true for the black lines in γ_1 and γ_3.

So we are left with the red arcs which together make C, and the blue arcs which together make $-C_1$ and $-C_2$ (notice that the blue arcs are travelling clockwise!)

All together, this shows that $\int_C \frac{1}{z^2+1}dz = \int_{C_1} \frac{1}{z^2+1}dz + \int_{C_1} \frac{1}{z^2+1}dz$. We now calculate these two integrals using CIF.

For the integral around C_1, let $f(z) = \frac{1}{z+i}$. Then $f(z)$ is analytic on $\mathbb{C} \setminus \{-i\}$, which is a domain containing C_1 and its inside. So:

$$\int_{C_1} \frac{1}{z^2+1}dz = \int_{C_1} \frac{f(z)}{z-i}dz = 2\pi i f(i) = \pi$$

And similarly, $\int_{C_2} \frac{1}{z^2+1}dz = -\pi$. So all together, $\int_C \frac{1}{z^2+1}dz = 0$.

This example suggests a general technique.

3.4.2 The Deformation Theorem

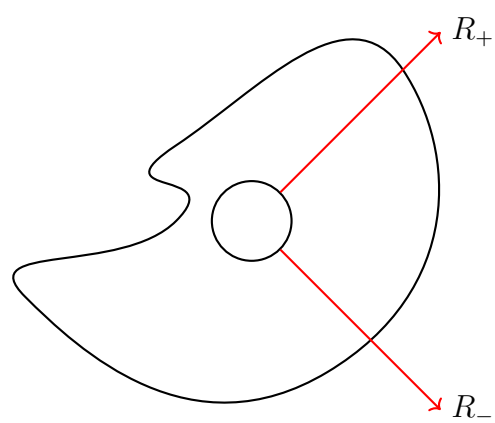
We can mimic the above technique for domains containing more poles. This will let us break up some quite complicated integrals into a handful of integrals which can be handled by other techniques (for example, the CIF.)

Theorem 3.4.2: The Deformation Theorem

Let D be a domain and $z_1, \ldots, z_n \in D$. Suppose γ is a piecewise smooth, positively oriented, simple closed curve in D such that the inside of γ is in D and each z_j is inside γ. Suppose that $f(z)$ is analytic on at least $D \setminus \{z_1, \ldots, z_n\}$. Further, suppose $r_1, \ldots, r_n > 0$ satisfy that $\{z \in \mathbb{C} | |z - z_j| \leq r_j\} \subset D$. Let C_j be the circle of radius r_j centered at z_j travelled once clockwise. Then:

$$\int_\gamma f(z)dz = \sum_{j=1}^{n} \int_{C_j} f(z)dz$$

Proof. We proceed by induction. Suppose $n = 1$. Fix $\theta \in (0, \pi)$. Consider the rays $R_+ = \{z_1 + re^{i\theta} | r \geq r_j\}$ and $R_- = \{z_1 + re^{-i\theta} | r \geq r_j\}$. We have something like the following picture:



Since z_j is inside γ, there exists r_+ and r_- which are the closest intersections of R_+ and R_- with γ to the point z_j. Let L_+ be the line segment from $z_0 + r_- e^{i\theta}$ to $z_0 + r_+ e^{i\theta}$ and L_- the line segment from $z_0 + r_j e^{-i\theta}$ to $z_0 + r_- e^{-i\theta}$.

These intersections divide γ into two segments: γ_1 and γ_2. Specifcally, γ_1 starts at the intersection of γ with R_+ and travels γ in the positive orientation until it hits the intersection with R_-. And γ_2 starts where γ intersects R_- and travels along γ until it hits R_+

Similarly, the circle is divided into two segments: the arc A_1 from the angle $-\theta$ to θ, and the arc A_2 from the angle θ to $2\pi - \theta$.

By our construction and the assumption that C_1 is inside γ, z_1 is not inside either of $\gamma_1 - L_- - A_2 + L_+$ or $\gamma_2 - L_+ - A_1 + L_-$. (Follow the picture to see why these are the curves we want.) These are each simple closed, positively oriented curves whose insides are in $D \setminus \{z_1\}$. As such, CIT gives that:

$$\int_{\gamma_1 - L_- - A_2 + L_+} f(z)dz = \int_{\gamma_2 - L_+ - A_1 + L_-} f(z)dz = 0$$

Adding these together, we get that:

$$\int_{\gamma_1} f(z)dz - \int_{L_-} f(z)dz - \int_{A_2} f(z)dz + \int_{L_+} f(z)dz + \int_{\gamma_2} f(z)dz - \int_{L_+} f(z)dz - \int_{A_1} f(z)dz - \int_{L_1} f(z)dz = 0$$

Simplifying gives that $\int_\gamma f(z)dz - \int_{C_1} f(z)dz = 0$, as desired.

Proceeding with the induction, suppose the claim is true for k points $z_1, \ldots, z_k$ where $k \leq n$.

Decompose γ as in the case for $n = 1$. However, in this case, we will have that $\gamma_1 - L_- - A_2 + L_+$ will now have (without loss of generality) $z_2, \ldots, z_k$ inside $\gamma_1 - L_- - A_2 + L_+$ for some $k \leq n$. The remaining $z_{k+1}, \ldots, z_n$ (if there are any) will be inside $\gamma_2 - L_+ - A_1 + L_-$ necessarily. Since $\{z_2, \ldots, z_k\}$ contains at most $n - 1$ points, the induction hypothesis applies to give:

$$\int_{\gamma_1 - L_- - A_2 + L_+} f(z)dz = \sum_{j=2}^{k} \int_{C_j} f(z)dz$$

and similarly the induction hypothesis gives us that:

$$\int_{\gamma_2 - L_+ - A_1 + L_-} f(z)dz = \sum_{j=k+1}^{n} \int_{C_j} f(z)dz$$

Adding these two together, as in the $n = 1$ case, gives:

$$\int_\gamma f(z)dz - \int_{C_1} f(z)dz = \sum_{j=2}^{n} \int_{C_j} f(z)dz$$

Rearranging gives the desired result. □

As an easy consequence of this, we can state a more general version of the Cauchy Integral Formula:

Corollary 3.4.1: Cauchy's Integral Formula (vII)

Suppose $f(z)$ is analytic on a domain D. Let γ be a piecewise smooth, positively oriented, simple closed curve in D whose inside is in D. Suppose z_0 is inside γ. Then:

$$\int_\gamma \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

This follows immediately from deforming γ to a circle and using our original CIF.

3.4.3 Poles of Higher Order - The Generalized Cauchy Integral Formula

Now that we know how to integrate curves surrounding a simple pole, or multiple simple poles, how do we handle integrating around higher order poles? For example, how do we find $\int_{|z|=1} \frac{e^z}{z^2} dz$? It turns out that we can generalize the CIF to handle this.

Theorem 3.4.3: The Generalized Cauchy Integral Formula (or the Cauchy Differentation Formula)

Suppose $f(z)$ is analytic on a domain D. Let γ be a piecewise smooth, positively oriented, simple closed curve in D whose inside is in D. Suppose z_0 is inside γ. Suppose $n > 0$. Then:

$$\int_\gamma \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)$$

The key to proving this is a result from multivariable calculus called the Leibniz Integral Rule.

Lemma 3.4.1. *Suppose $f(w,z)$ is continuous in both z and w on some region R such that if $(w_0, \gamma(t)) \in R$ for all t whenever there exists $(w_0, z) \in R$. Further, suppose that $f_w(w,z)$ is continuous in both z and w. Then:*

$$\frac{d}{dw} \int_\gamma f(w,z) dz = \int_\gamma \frac{\partial}{\partial w} f(w,z) dz$$

We will not be proving this. We move on to the proof of our theorem:

Proof. We proceed by induction. For $n = 1$, the claim holds by the regular Cauchy Integral Formula.

Suppose the claim holds for some n. Let $g(z_0, z) = \frac{f(z)}{(z-z_0)^n}$. Notice that since z_0 is inside γ and z is on γ, that $g(z_0, z)$ is continuous in z_0 and z (we can actually take z close to γ, since z_0 has some positive distance between it and γ). Further:

$$\frac{\partial}{\partial z_0} g(z_0, z) = \frac{nf(z)}{(z-z_0)^{n+1}}$$

is also continuous on the same region, for the same reason. As such, Leibniz's rule gives:

$$\begin{aligned}
\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} dz &= \int_\gamma \frac{1}{n} \frac{\partial}{\partial z_0} g(z_0, z) dz \\
&= \frac{1}{n} \frac{d}{dz_0} \int_\gamma g(z_0, z) dz \\
&= \frac{1}{n} \frac{d}{dz_0} \int_\gamma \frac{f(z)}{(z-z_0)^n} dz
\end{aligned}$$

However, our induction hypothesis gives that $\int_\gamma \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)$. So:

$$\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{1}{n}\frac{d}{dz_0}\frac{2\pi i}{(n-1)!}f^{(n-1)}(z_0)$$
$$= \frac{2\pi i}{n!}f^{(n)}(z_0)$$

□

Note that at no point did we assume that the derivatives $f^{(n)}(z_0)$ exist. In fact, the Leibniz rule gives us that they exist, since they're equal to these integrals which do exist! As a corollary:

Corollary 3.4.2: Holomorphic Functions are Smooth

If f is holomorphic on a domain D, then f is infinitely differentiable (otherwise known as smooth) on D.

Let's finish off with an example of using the generalized CIF.

Example 3.4.5

Let γ be the circle of radius 2 centered at 0 travelled twice clockwise. Find $\int_\gamma \frac{\sin(z)}{(z^2+1)^2}dz$.

This function has two double poles, $z = \pm i$. Let C be the circle of radius 2 centered at 0 travelled once clockwise. Then:

$$\int_\gamma \frac{\sin(z)}{(z^2+1)^2}dz = -2\int_C \frac{\sin(z)}{(z^2+1)^2}dz$$

By the deformation theorem:

$$\int_C \frac{\sin(z)}{(z^2+1)^2}dz = \int_{|z-i|=1} \frac{\sin(z)}{(z^2+1)^2}dz + \int_{|z+i|=1} \frac{\sin(z)}{(z^2+1)^2}dz$$

For the circle centered at i, we let $f(z) = \frac{\sin(z)}{(z+i)^2}$. Then $\frac{\sin(z)}{(z^2+1)^2} = \frac{f(z)}{(z-i)^2}$. Since $f(z)$ is analytic on $\mathbb{C} \setminus \{-i\}$, we can apply the generalized CIF to get:

$$\int_{|z-i|=1} \frac{\sin(z)}{(z^2+1)^2}dz = 2\pi i f'(i)$$

Now, $f'(z) = \frac{\cos(z)(z+i)^2 - 2\sin(z)(z+i)}{(z+i)^4}$, so:

$$f'(i) = \frac{\frac{e^{-1}+e}{2}(2i)^2 - 2\frac{e^{-1}-e}{2i}(2i)}{16} = \frac{-4(e^{-1}+e) - 4(e^{-1}-e)}{32} = -\frac{1}{4e}$$

And so $\int_{|z-i|=1} \frac{\sin(z)}{(z^2+1)^2}dz = -\frac{\pi i}{2e}$.

For the circle centered at $-i$, we follow the same procedure with $g(z) = \frac{\sin(z)}{(z-i)^2}$. We find that:

$$\int_{|z+i|=1} \frac{\sin(z)}{(z^2+1)^2} dz = 2\pi i g'(-i)$$

And we calculate:

$$g'(-i) = \frac{-4\cos(-i) + 4i\sin(-i)}{16} = -4\frac{e^{i*i}}{16} = -\frac{1}{4e}$$

As such, $\int_{|z+i|=1} \frac{\sin(z)}{(z^2+1)^2} dz = -\frac{\pi i}{2e}$.

All together, $\int_\gamma \frac{\sin(z)}{(z^2+1)^2} dz = -2(-\frac{\pi i}{2e} - \frac{\pi i}{2e}) = \frac{2\pi i}{e}$.

3.5 Liouville's Theorem

For the moment, we take a small detour from our roadmap to investigate a peculiar property of entire functions. Entire functions behave quite differently than we're used to. For example, our intuition says that the function $f(z) = \sin(z)$ should be bounded. After all, for $x \in \mathbb{R}$, $-1 \leq \sin(x) \leq 1$. However, we've seen that $\sin(z)$ is not a bounded function!

This is an example of a much more general phenomenon:

Theorem 3.5.1: Liouville's Theorem

Every bounded, entire function is constant.

Proof.

Proof idea: To show $f(z)$ is constant, we can show that $f'(z) = 0$ for all $z \in \mathbb{C}$.

To connect $f'(z)$ to the fact that $f(z)$ is bounded, we go through Cauchy's integral formula, which connects $f'(z)$ to $f(z)$.

Proof: Suppose $f(z)$ is bounded and entire, but is non-constant. Let $M \in \mathbb{R}$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$.

Let γ_R be the circle of radius of circle R, centered at a, travelled once counter-clockwise. Since $f(z)$ is entire, Cauchy's integral formula tells us that:

$$2\pi i f'(a) = \int_{\gamma_R} \frac{f(z)}{(z-a)^2} dz$$

We would like to show this integral is 0. However, we don't have any way to calculate it. Instead, let's estimate:

$$|2\pi i f'(a)| = \left| \int_{\gamma_R} \frac{f(z)}{(z-a)^2} dz \right| \leq \int_{\gamma_R} \left| \frac{f(z)}{(z-a)^2} \right| dz = \int_{\gamma_R} \frac{|f(z)|}{|z-a|^2} dz$$

Now, since z is on γ_R, which is the circle of radius R centered at a, we see that $|z-a| = R$. So:

$$|2\pi i f'(a)| \leq \int_{\gamma_R} \frac{|f(z)|}{R^2} dz$$

Now, by M-L estimation (theorem 3.2.7), we see that:

$$|2\pi i f'(a)| \leq \frac{2\pi R M}{R^2} = \frac{2\pi M}{R}$$

However, notice this is true for all R. So by the squeeze theorem:

$$0 \leq |2\pi i f'(a)| \leq \lim_{R\to\infty} \frac{2\pi M}{R} = 0$$

As such, $2\pi i f'(a) = 0$. So $f'(a) = 0$. Since a was arbitrary, we have shown that $f'(z) = 0$ for all z, so that $f(z)$ is a constant function.

□

This is a fairly strong statement. It is also a favourite for coming up with interesting proof questions on tests. Let's see an example:

Example 3.5.1

Suppose $f(z)$ is entire and $f(z) \neq kz$ for any k. (Meaning the function $f(z)$ is not equal to the function kz.) Then there exists $w \in \mathbb{C}$ with $|f(w)| \leq |w|$.

Suppose $|z| < |f(z)|$ for all $z \in \mathbb{C}$. Consider the function $g(z) = \frac{z}{f(z)}$. Since $0 \leq |z| < |f(z)|$ for all $z \in \mathbb{C}$, we have that $f(z) \neq 0$ for all z. This means that $g(z)$ is entire.

However, we also know that $|g(z)| = \frac{|z|}{|f(z)|} < 1$ for all $z \in \mathbb{C}$. So by Liouville, $g(z)$ is constant. In particular, $\frac{z}{f(z)} = k$ for some $k \in \mathbb{C}$. I.e., $f(z) = kz$. A contradiction. Therefore, $|f(z)| \leq |z|$ for some $z \in \mathbb{C}$.

We can also use Liouville's theorem to prove

Theorem 3.5.2: The Fundamental Theorem of Algebra

Every non-constant complex polynomial has a complex root.

Proof.

Proof Idea: To show $p(z)$ has roots, we look at $\frac{1}{p(z)}$. We use Liouville to show that if $p(z)$ has no roots, then this new function is actually constant. That contradicts that $p(z)$ is non-constant.

Proof: Let $p(z)$ be a non-constant complex polynomial. We proceed by contradiction. Assume $p(z) \neq 0$ for all z.

Consider $f(z) = \frac{1}{p(z)}$. Since $p(z)$ is entire and non-zero, $f(z)$ is also entire. We claim that $f(z)$ is bounded, so that we can use Liouville's theorem.

We will handle this in two pieces: show $p(z)$ is bounded on some large closed circle, and show it's bounded outside that circle as well.

However, we need to figure out what this circle actually is. To do that, we're going to look at $\lim_{z\to\infty} f(z)$.

Limit: Let $p(z) = a_n z^n + \cdots + a_1 z + a_0$. Then using the triangle inequality, we find that $|p(z)| \geq |a_n||z^n| - (|a_{n-1}||z|^{n-1} + \cdots + |a_0|)$. Suppose $|z| = R$. Then:

$$\lim_{z\to\infty} |p(z)| \geq \lim_{z\to\infty} |a_n|R^n - (|a_{n-1}|R^{n-1} + \cdots + |a_0|) = \infty$$

This tells us that $\lim_{z\to\infty} p(z) = \infty$, and so $\lim_{z\to\infty} f(z) = 0$.

Circle We can use this limit fact to find a large circle such that $f(z)$ is bounded outside that circle. Remember, $\lim_{z\to\infty} f(z) = L$ means that:

$$\forall \varepsilon > 0, \exists R, \text{ such that } |z| > R \Rightarrow |f(z) - L| < \varepsilon$$

Choose $\varepsilon = 1$. Since $\lim_{z\to\infty} f(z) = 0$, there exists some radius R such that when $|z| > R$, we have $|f(z) - 0| < 1$. I.e., $|f(z)| < 1$.

So $f(z)$ is bounded outside the disc $|z| \leq R$, by definition.

Inside the Circle So what happens for $|z| \leq R$?

We're going to use a familiar result: the Extreme Value Theorem. However, proving the complex versionof this theorem is actually fairly technical. As such, we will not be presenting a proof here. For completeness, it will appear in appendix A.1.

Lemma 3.5.1. *Suppose $f(z)$ is a continous complex function. If C is a closed and bounded subset of $\mathbb{C}$, then $f(C) = \{f(z)|z \in C\}$ is bounded.*

We know that $\{z||z| \leq R\}$ is closed and bounded (by an example in the appendix), and that $f(z)$ is continuous (since it is entire). So by EVT, $f(z)$ is also bounded on $|z| \leq R$. Say $|f(z)| \leq M$.

All Together So, if $z \in \mathbb{C}$, then $|z| \leq R$ or $|z| > R$. If $|z| \leq R$, then $|f(z)| \leq M$. And if $|z| > R$, then $|f(z)| < 1$. So $|f(z)| \leq \max\{1, M\}$.

As such, $f(z)$ is a bounded function. It is also entire. And so is constant.

But then $p(z) = \frac{1}{f(z)}$ is also constant, contradicting that $p(z)$ is non-constant. Therefore, $p(z)$ has roots.

$\square$

4 Power and Laurent Series

Last chapter, we laid out an integration road map. Unfortunately, we can't handle removable or essential singularities without further investigating the properties of complex functions. The key tool to understanding these kinds of singularities is the notion of a Laurent series.

Well, if we're going to be looking at power series, we had best first talk about sequences and series.

4.1 Sequences and Series

Definition 4.1.1: Sequence

A sequence in $\mathbb{C}$ is an infinite list of complex numbers $a_k, a_{k+1}, \ldots$. We will write this as $(a_n)_{n=k}^{\infty}$.

We say that the sequence $(a_n)_{n=k}^{\infty}$ converges to L, or that $\lim_{n\to\infty} a_n = L$, or even that $a_n \to L$, if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } n > N \implies |a_n - L| < \varepsilon$$

If a sequence does not converge to any limit L, then it diverges.

We aren't going to spend a lot of time discussing sequences. We only care about them in the context of series.

Definition 4.1.2: Series

Let $(a_n)_{n=k}^{\infty}$. Define a new sequence $(S_n)_{n=k}^{\infty}$ by:

$$S_n = \sum_{j=k}^{n} a_j$$

This is called the nth partial sum of the sequence $(a_n)_{n=k}^{\infty}$. The infinite series $\sum_{n=k}^{\infty} a_n$ is defined to be $\sum_{n=k}^{\infty} a_n = \lim_{n\to\infty} S_n$, if the sequence converges. In that case, we say the sequence converges as well.

Let's look at a couple of examples of convergence.

Example 4.1.1

The **geometric series** $\sum_{n=0}^{\infty} z^n$ converges to $\frac{1}{1-z}$ if $|z| < 1$ and diverges if $|z| > 1$.

Consider the nth partial sum:

$$S_n = 1 + z + z^2 + \cdots + z^n$$

This is a finite geometric series, which is equal to $\frac{1-z^{n+1}}{1-z}$. To see this, notice that:

$$\begin{aligned}(1-z)(1+z+\cdots+z^n) &= (1-z)+z(1-z)+z^2(1-z)+\cdots+z^n(1-z)\\ &= 1-z+z-z^2+z^2-\cdots-z^n+z^n-z^{n+1}\\ &= 1-z^{n+1}\end{aligned}$$

So for $z \neq 1$, $S_n = \frac{1-z^{n+1}}{1-z}$. We now take the limit of this sequence:

$$\lim_{n\to\infty} \frac{1-z^{n+1}}{1-z} = \frac{1-\lim_{n\to\infty} z^{n+1}}{1-z}$$

If $|z| < 1$, notice that $|z^{n+1}| \to 0$ (while we could show this formally, this is something you saw in first year calc.) Furthermore, if $|z| > 1$, then $|z^{n+1}| \to \infty$. This tells us that if $|z| < 1$ that $z^{n+1} \to 0$ and if $|z| > 1$ that $z^{n+1} \to \infty$. As such:

$$\lim_{n\to\infty} S_n = \begin{cases} \frac{1}{1-z}, & |z| < 1 \\ \infty, & |z| > 1 \end{cases}$$

This example is going to be very useful. Make sure you know this formula.

For a lot of series, actually calculating the limit can be quite hard. For example, how would you show that $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$, or that $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} = \sin(1)$?

For the moment, let's instead focusing on determining if a series converges at all. We begin by recalling a couple of convergence results that you should have already seen for $\mathbb{R}$.

Theorem 4.1.1: The Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=k}^{\infty} a_n$ diverges.
The converse is false.

Proof. We prove the contrapositive instead. Suppose $\sum_{n=k}^{\infty} a_n$ converges.

Notice that $a_n = S_n - S_{n-1}$. As such, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$. Notice that the sequences $(S_n)_{n=k}^{\infty}$ and $(S_{n-1})_{n=k+1}^{\infty}$ have the same limit. Indeed, these are actually the same sequence, just indexed differently! So:

$$\lim_{n\to\infty} a_n = \sum_{n=k}^{\infty} a_n - \sum_{n=k}^{\infty} a_n = 0$$

For the converse, we claim that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. The idea is to write the sum as:

$$1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{<\frac{1}{4}+\frac{1}{4}=\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{<\frac{4}{8}=\frac{1}{2}}+\dots$$

More generally, notice that $\sum_{n=2^k+1}^{2^{k+1}} \frac{1}{n} > \sum_{n=2^k+1} 2^{k+1}\frac{1}{2^{k+1}} = \frac{1}{2}$. As such:

$$S_{2^{m+1}} = \sum_{n=1}^{2^{m+1}} \frac{1}{n} = 1+\frac{1}{2}+\sum_{k=1}^{m}\sum_{n=2^k+1}^{2^{k+1}} \frac{1}{n} \geq 1+\frac{1}{2}+\frac{m}{2}$$

Therefore, $\lim_{n\to\infty} S_n = \lim_{m\to\infty} 1+\frac{m+1}{2} = \infty$. So the series diverges.

This is called the harmonic series, and is the classic example of a series whose terms go to 0 be which diverges anyway. □

This is not the only example of a series whose terms go to 0 but which diverges. There are plenty of interesting examples of this behavior. For example, $\displaystyle\sum_{p \text{ prime}} \frac{1}{p}$ also diverges, although much more slowly. While outside the scope of this course, this is traditionally shown using the Prime Number Theorem, which is equivalent to saying that the nth prime p_n is approximately $n\ln(n)$ for n very large.

Another way to tell if a series is convergent is if it is absolutely convergent.

Definition 4.1.3: Absolutely Convergent Series

A series $\sum_{n=k}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=k}^{\infty} |a_n|$ converges.

Theorem 4.1.2

Absolutely convergent series converge.

Proof. See A.2.1 in the appendices for a proof. It is a technical proof, using the notion of Cauchy sequences. I have included it for completeness's sake. □

With this new notion, we can prove a very powerful convergence test.

Theorem 4.1.3: D'Alembert's Ratio Test

Suppose $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$.

- If $L<1$, then $\sum_{n=k}^{\infty} a_n$ converges absolutely. (And therefore converges.)
- If $L>1$, the series diverges.

Proof. Suppose $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$.

Suppose $L < 1$. Choose $\varepsilon > 0$ so that $L < L + \varepsilon < 1$. Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, there exists $N \in \mathbb{N}$ with $N > k$ such that if $n \geq N$, then $\left|\frac{a_{n+1}}{a_n}\right| < L + \varepsilon$.

As such, $|a_{N+j}| < (L+\varepsilon)|a_{N+j-1}|$ for all $j \geq 1$. We can use this iteratively to conclude that $|a_{N+j}| < (L+\varepsilon)^j |a_N|$.

We see then that $\sum_{n=k}^{\infty} |a_n| < \sum_{n=k}^{N-1} |a_n| + \sum_{j=N}^{\infty} (L+\varepsilon)^j |a_N|$. Since $L + \varepsilon < 1$, this geometric series converges.

Now, the Comparison Theorem (from first year calculus) tells us that if $0 \leq a_n \leq b_n$ and $\sum_{n=k}^{\infty} b_n$ converges, then so does $\sum_{n=k}^{\infty} a_n$. So, by the Comparison Theorem, $\sum_{n=k}^{\infty} |a_n|$ converges.

Therefore, $\sum_{n=k}^{\infty} a_n$ converges absolutely.

Now, supposing that $L > 1$, we choose ε so that $1 < L - \varepsilon < L$. Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, there exists $N \in \mathbb{N}$ with $N > k$ such that if $n \geq N$, then $\left|\frac{a_{n+1}}{a_n}\right| > L - \varepsilon$.

In this case, we know that for $j \geq 0$, $|a_{N+j}| > (L-\varepsilon)^j |a_N|$. However, as $j \to \infty$, $(L_\varepsilon)^j \to \infty$. As such, $\lim_{n\to\infty} a_n = \infty$. By the Divergence Test, $\sum_{n=k}^{\infty} a_n$ diverges.

$\square$

This is going to be an extremely useful test for us. In fact, when dealing with power series (as we will be very shortly), this test is the most useful.

Example 4.1.2

The series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all n.

To see this, we apply the ratio test. We need to be careful here. Remember that in the ratio test, a_n represents the nth term of the series. It does not represent the coefficient of z^n.

So, in this case, $a_n = \frac{z^n}{n!}$. And therefore:

$$\lim_{n\to\infty} \left| \frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}} \right| = \lim_{n\to\infty} \frac{|z|}{n+1} = 0$$

Since $L = 0 < 1$ for all $z \in \mathbb{C}$, we see that the series converges absolutely for each $z \in \mathbb{C}$.

Example 4.1.3

Find an $R \geq 0$ such that if $|z| < R$, then $\sum_{n=0}^{\infty} \frac{z^n}{2^n}$ converges, and if $|z| > R$, then the sum diverges.

This turns out to follow right away from the ratio test. We apply:

$$\lim_{n\to\infty} \left| \frac{\frac{z^{n+1}}{2^{n+1}}}{\frac{z^n}{2^n}} \right| = \lim_{n\to\infty} \frac{|z|}{2}$$

Now, if $\frac{|z|}{2} < 1$, the ratio test tells us that the series converges. And if $\frac{|z|}{2} > 1$, the series diverges. In other words, $R = 2$ works.

4.2 Power Series

Now that we've discussed series and developed the relevent tools, we can now talk about power series.

Definition 4.2.1: Power series

A power series centered at z_0 is a function $f(z)$ given by:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

The domain of this function is every point where this series converges, and contains at least z_0.

For some function $g(z)$, we say that $g(z)$ is represented by the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ on a domain D if $g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ on D.

We've already seen one example of a power series.

Example 4.2.1

$\frac{1}{1-z}$ is represented by the power series $\sum_{n=0}^{\infty} z^n$ on $B_1(0)$.

One of the most crucial facts moving forward is that holomorphic functions can be represented by power series. This is a drastic difference between the complex and real theory. There are infinitely differentiable functions over $\mathbb{R}$ which cannot be meaningfully represented by power series. That is not the case over $\mathbb{C}$.

Theorem 4.2.1

Suppose $f(z)$ is holomorphic on a domain D, and $z_0 \in D$. Then there exists some $R > 0$ such that if $|z - z_0| < R$, then:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

Proof. Suppose $f(z)$ is holomorphic on D. Let $R > 0$ such that $B_R(z_0) \subset D$.

Let $w \in B_R(z_0)$. Our goal is to show that $f(w) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (w - z_0)^n$.

Choose $r \in \mathbb{R}$ such that $|w - z_0| < r < R$. This can be done since $w \in B_R(z_0) \implies |w - z_0| < R$. Let γ_r be the circle of radius r centered at z_0, travelled once counterclockwise. By the Cauchy Integral Formula:

$$f(w) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z - w} dz$$

For the moment, let us consider $\frac{1}{z-w}$. We can rewrite this as:

$$\frac{1}{z-w} = \frac{1}{(z-z_0)-(w-z_0)} = \frac{1}{(z-z_0)}\frac{1}{1-\frac{w-z_0}{z-z_0}}$$

Now, since $|w-z_0| < r$ and $|z-z_0| = r$ (since z is on γ_r), we know that $\left|\frac{w-z_0}{z-z_0}\right| < 1$. So, our geometric series formula gives that:

$$\frac{1}{z-w} = \frac{1}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^n = \sum_{n=0}^{\infty}\frac{(w-z_0)^n}{(z-z_0)^{n+1}}$$

And therefore, $f(w) = \frac{1}{2\pi i}\int_{\gamma_r}\sum_{n=0}^{\infty}(w-z_0)^n\frac{f(z)}{(z-z_0)^{n+1}}dz$. By uniform convergence of the series (see theorem A.2.3 in the appendices for the very technical proof), we have that:

$$\begin{aligned}
f(w) &= \frac{1}{2\pi i}\int_{\gamma_r}\sum_{n=0}^{\infty}(w-z_0)^n\frac{f(z)}{(z-z_0)^{n+1}}dz \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\gamma_r}(w-z_0)^n\frac{f(z)}{(z-z_0)^{n+1}}dz \\
&= \sum_{n=0}^{\infty}\frac{(w-z_0)^n}{2\pi i}\int_{\gamma_r}\frac{f(z)}{(z-z_0)^{n+1}}dz \\
&\overset{CIF}{=} \sum_{n=0}^{\infty}\frac{(w-z_0)^n}{2\pi i}\frac{2\pi i f^{(n)}(z_0)}{n!} \\
&= \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(w-z_0)^n
\end{aligned}$$

as desired. Since w was chosen arbitrarily in $B_R(z_0)$, the power series expansion holds on $B_R(z_0)$.

□

This theorem is sometimes stated very simply as "holomorphic functions are analytic". We have used holomorphic and analytic interchangeably throughout this text. However, f analytic really means that the function can be described as a convergent power series. We have just shown that holomorphic functions actually are analytic, they can be described by power series.

Let's work out an example by calculating an example of a power series.

Example 4.2.2

Consider $f(z) = e^z$. Find the power series expansion for e^z centered at z_0. For which $z \in \mathbb{C}$ is it valid?

Well, $f^{(n)}(z) = e^z$, and so the power series expansion of e^z at z_0 is:

$$\sum_{n=0}^{\infty} \frac{e^{z_0}}{n!}(z-z_0)^n$$

Now, the theorem above was valid on $B_R(z_0)$ where $B_R(z_0)$ is contained in a domain where $f(z)$ is holomorphic. However, e^z is entire, and so all R satisfy this condition. As such, the power series expansion is valid on $B_R(z_0)$ for any R. Since for each $w \in \mathbb{C}$, there exists R so that $w \in B_R(z_0)$, it follows that this power series expansion is valid on $\mathbb{C}$.

Knowing that holomorphic functions can be described by convergent power series raises a couple of other questions:

- How can we tell more generally when a power series converges?
- Holomorphic functions have power series. Are power series holomorphic?

Let's begin by discussing the first of these.

Definition 4.2.2: Radius of Convergence

The radius of convergence of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is:

$$R = \max\{r \in \mathbb{R} | \text{ the series converges on } B_r(z_0)\}$$

If no such R exists, then we say $R = 0$. And if the series converges on all balls $B_r(z_0)$, we say $R = \infty$.

Some things to keep in mind here: if we're looking at power series of a holomorphic function, the radius of convergence depends on the center of the series. It's not usually a one size fits all situation. There is one situation where that's not the case though:

Example 4.2.3

Suppose $f(z)$ is entire. Then the power series expansion for $f(z)$ at z_0 has radius of convergence $R = \infty$ for any z_0.

However, theorem 4.2.1 tells us that if $f(z)$ is analytic on $B_r(z_0)$, then its power series expansion at z_0 converges to the value of the function on $B_r(z_0)$. As such, $R \geq r$. But since $f(z)$ is analytic on $B_r(z_0)$ for all r, this means that $R > r$ for all r. No real number satisfies this, and so $R = \infty$.

Alright, this helps us figure out the radius of convergence in some situations. But what if we don't know what function the power series gives? Is there some way to find R in that case?

Theorem 4.2.2: Ratio Test for Power Series

Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ where $L \in [0, \infty]$. Then:

- If $L = 0$, the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence $R = \infty$.

- If $L = \infty$, the series has radius of convergence 0.
- If $L \in (0, \infty)$, the series has radius of convergence $R = \frac{1}{L}$.

Proof. We apply the usual ratio test to the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$.

We know the series always converges when $z = z_0$, so we only need to consider when $z \neq z_0$. We compute:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(z - z_0)^{n+1}}{a_n(z - z_0)^n} \right| = L|z - z_0|$$

Now, if $L|z - z_0| < 1$, this series converges absolutely. And when $L|z - z_0| > 1$, it diverges. So, we consider our cases:

- If $L = 0$, then $L|z - z_0| = 0 < 1$ for all $z \in \mathbb{C}$. As such, the series converges everywhere and $R = \infty$.
- If $L = \infty$, we have that $\lim_{n\to\infty} \left| \frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n} \right| > 1$ for any $z \neq z_0$. As such, the series diverges on $\mathbb{C} \setminus \{z_0\}$ and $R = 0$.
- If $L \in (0, \infty)$, the the series converges when $|z - z_0| < \frac{1}{L}$ and diverges when $|z - z_0| > \frac{1}{L}$. As such, $B_{\frac{1}{L}}(z_0)$ is the largest open ball on which the series converges, so $R = \frac{1}{L}$.

$\square$

The helpful mnemonic here is that $\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n} \right|$, being aware that this is not strictly true if the limit is 0 or ∞.

Example 4.2.4

Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n z^n}{2^n n^3}$.

Well, we compute:

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{2^{n+1}(n+1)^3}}{\frac{(-1)^n}{2^n n^3}} \right| = \lim_{n\to\infty} \frac{n^3}{2(n+1)^3} = \frac{1}{2}$$

As such, $R = 2$.

Let's look at a bit of a more complicated example.

Example 4.2.5

$\cos(z)$ has power series expansion at $z_0 = 0$:

$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$$

Now, since $\cos(z)$ is entire, we know this has radius of convergence $R = \infty$. Out of curiousity, is it possible to use this formula to find this?

Well, we have:

$$\cos(z) = 1 + 0z - \frac{1}{2}z^2 + 0z^3 + \frac{1}{24}z^4 + 0z^5 + ...$$

In this case, $a_{2n} = \frac{(-1)^n}{(2n)!}$ and $a_{2n+1} = 0$. So, the sequence $b_n = \frac{a_{n+1}}{a_n}$ has $b_{2n} = 0$ and b_{2n+1} undefined! So we can't take $\lim_{n\to\infty} |b_n|$, since the sequence isn't defined for all n.

How can we fix this? One way is to set $w = z^2$. Then:

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n w^n}{(2n)!}$$

This series isn't missing any terms, so we can use the ratio test with $a_n = \frac{(-1)^n}{(2n)!}$. We find that this series converges when $|w| < R$ where:

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(2(n+1))!}}{\frac{(-1)^n}{(2n)!}} \right| = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0$$

So, this series (in terms of w!) has radius of convergence $R = \infty$. Since it converges for all w, it also converges for all z.

In this case, it didn't matter that we had $|w| < R$ vs. $|z| < R$. But, for example, if we found that the series in terms of w had radius of convergence 4, we would have $|w| < 4$. But $w = z^2$, so this gives $|z^2| < 4$ or $|z| < 2$. The moral: be careful.

Can we find the radius of convergence of the power series for some function without actually computing the power series?

Example 4.2.6

Consider $f(z) = \frac{1}{z^2-1}$. This function is analytic on $\mathbb{C} \setminus \{\pm 1\}$, so it has a power series expansion at each of those points by theorem 4.2.1. Let's find the radius of convergence of this series at $z_0 = 3$.

Rather than actually compute the power series, notice that the theorem tells us that if $f(z)$ is analytic on $B_r(z_0)$. then:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

for any $z \in B_r(z_0)$. As such, if $f(z)$ is analytic on $B_r(z_0)$, then this power series must have radius of convergence $R \geq r$.

In our case, notice that $B_2(3)$ is the largest ball centered at 3 on which $f(z)$ is analytic. As such, we have that $R \geq 2$.

In fact, $R = 2$! But to see this, we're going to need to develop some more techniques.

To finish this example, we prove (or state) some theorems that will help us determine the radius of convergence.

Theorem 4.2.3

Suppose $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. Let $w \in \mathbb{C}$.

If $f(z)$ converges at $z = w$, then $f(z)$ converges at z for all $z \in \mathbb{C}$ with $|z - z_0| < |w - z_0|$. And if $f(z)$ diverges at $z = w$, it diverges for all $z \in \mathbb{C}$ with $|z - z_0| > |w - z_0|$.

Before we prove this, let's talk about what this means. If this power series has radius of convergence R, this is saying that the series diverges for all $z \in \mathbb{C}$ with $|z - z_0| > R$. Why? Well, if it converges at such a z, then it converges on $B_{|z-z_0|}(z_0)$ which is a bigger ball than $B_R(z_0)$. Since $B_R(z_0)$ is the largest ball on which the series converges, this can't happen. So we get the useful corollary:

Corollary 4.2.1

If the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ has radius of convergence R, then the series may only converge on $\{z \in \mathbb{C} | |z - z_0| \leq R\}$. It diverges if $|z - z_0| > R$ and may either converge or diverge when $|z - z_0| = R$.

Proof. We now prove the theorem. The idea is to try to write the series as being close to a geometric series.

Suppose $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges at $z = w$. Suppose $|z' - z_0| < |w - z_0|$. Then:

$$\sum_{n=0}^{\infty} a_n(z' - z_0)^n = \sum_{n=0}^{\infty} a_n(w - z_0)^n \frac{(z' - z_0)^n}{(w - z_0)^n}$$

Let $\rho = \frac{z'-z_0}{w-z_0}$. Note that $|\rho| < 1$ since $|z' - z_0| < |w - z_0|$.

Now, since $\sum_{n=0}^{\infty} a_n(w - z_0)^n$ converges, we know that $\lim_{n\to 0} a_n(w - z_0)^n = 0$ by the divergence test. As such, $\exists M \in \mathbb{R}$ so that for all $n \in N$, $|a_n(w - z_0)^n| < M$.

Now, we have that $|a_n(z' - z_0)^n| \leq M\rho^n$. Since $\sum_{n=0}^{\infty} M\rho^n$ converges, the comparison test for real series tells us that $\sum_{n=0}^{\infty} |a_n(z' - z_0)^n|$ converges as well.

Therefore, $\sum_{n=0}^{\infty} a_n(z' - z_0)^n$ converges absolutely, and hence converges. $\square$

Next, we state a very important but very technical result. The proof will appear in the appendices.

Theorem 4.2.4: Power Series are differentiable

Suppose $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ has radius of convergence $R > 0$. Then $g(z) = \sum_{n=1}^{\infty} na_n(z - z_0)^{n-1}$ also has radius of convergence R and $f'(z) = g(z)$.

To put this more plainly, the derivative of a power series is the term by term derivative. This also lets us prove something about primitives of power series.

Theorem 4.2.5: Power Series have primitives

Suppose $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence $R > 0$. Then $F(z) = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}$ also has radius of convergence R and $F'(z) = f(z)$.

Proof. To begin, we show that they have the same radius of convergence. We know that $f(z)$ has radius of convergence:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

if this limit exists. (A more precise version of this argument would use the concept of lim sup which always exists.)

If $F(z)$ has radius of convergence R_F, then:

$$R_F = \lim_{n\to\infty} \left| \frac{(n+2)a_n}{na_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = R$$

So they have the same radius of convergence. Then, by theorem 4.2.4, $F'(z) = f(z)$. □

Let's finish off our example from last class before we talk about other ways to use these results.

Example 4.2.7

We know that the power series for $\frac{1}{z^2-1}$ centered at $z_0 = 3$ has radius of convergence $R \geq 2$.

Suppose $R > 2$. This tells us that the power series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges on $B_R(3)$, and hence is differentiable there. This implies the series is continuous at $z = 1$.

Now, we know that for $|z - z_0| < 2$ that $f(z) = \frac{1}{z^2-1}$, and so we consider the limit as $z \to 1$ along the real axis. Set $z = r$ with $r \in (1, \infty)$. Then:

$$f(1) = \lim_{z\to 1} f(z) = \lim_{r\to 1^+} f(r) = \lim_{r\to 1^+} \frac{1}{r^2-1} = \infty$$

This is a contradiction, since we know that $f(1)$ is defined (and therefore not ∞.) So we cannot have $R > 2$, leaving us with $R = 2$.

Does this apply more broadly? What about other functions?

Theorem 4.2.6

Suppose $f(z)$ is analytic on $\mathbb{C}\backslash\{z_1, z_2, \dots\}$ where each $z_j \in D$ and is either a pole or essential singularity for $f(z)$.

Let $z_0 \in D$ and $z_0 \neq z_j$ for all j. Then the radius of convergence for the power series expansion of

$f(z)$ centered at z_0 is:

$$R = \min\{|z_0 - z_j|\}$$

Proof. The proof proceeds analagously to the previous example. We know that $f(z)$ is analytic on $B_R(z_0)$, since this ball contains none of the isolated singularities of $f(z)$.

However, since $\lim_{z \to z_j} f(z)$ does not exist, we cannot extend the power series to be valid beyond any of the z_j. Since $B_R(z_0)$ is the largest ball containing none of the z_j, this is the largest ball on which the series converges. □

What other ways can we use these results? They allow us to create new series without much fuss, so let's see what this can give us.

Example 4.2.8

Find the radius of convergence for $\sum_{n=1}^{\infty} nz^{n-1}$ and $\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}$. What functions are these?

We recognize that nz^{n-1} is the derivative of z^n. So:

$$\sum_{n=1}^{\infty} nz^{n-1} = \sum_{n=1}^{\infty} \frac{d}{d\,z} z^n = \frac{d}{dz} \frac{1}{1-z} = \frac{1}{(1-z)^2}$$

And theorem 4.2.4 tells us that this series has the same radius of convergence as the power series for $\frac{1}{1-z}$ centered at $z_0 = 0$, which is $R = 1$.

Similarly, we recognize that $\frac{z^{n+1}}{n+1}$ is a primitive for z^n, and so $\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}$ is a primitive for $\frac{1}{1-z}$. As such, it also has radius of convergence $R = 1$ and we have:

$$\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} = -\log_0(1-z) + C$$

for some logarithm $\log_0(z)$. Luckily, evaluating at $z = 0$ gives $0 = -\log_0(1) + C$. We can therefore choose $\log_0(z) = \mathrm{Log}(z)$ and $C = 0$.

Example 4.2.9

Find $\sum_{n=2}^{\infty} \frac{(-1)^n}{2^n(n^2-n)}$.

Well, this isn't a power series. However, this is the power series:

$$\sum_{n=2}^{\infty} \frac{z^n}{n(n-1)}$$

evaluated at $z = \frac{-1}{2}$. So, we need to investigate this power series.

To begin, we need to figure out how this series was built. The division by n and $n-1$ signals to me that this was built by taking primitives. The fact that we have two of them suggests we took the primitive of the primitive. To see this, let's reindex. Set $m = n-2$. Then we get:

$$\sum_{n=2}^{\infty} \frac{z^n}{n(n-1)} = \sum_{m=0}^{\infty} \frac{z^{m+2}}{(m+2)(m+1)}$$

Now, this is the primitive of:

$$\sum_{m=0}^{\infty} \frac{z^{m+1}}{(m+1)}$$

which the previous example tells us is $-\text{Log}(1-z)$. As such, this series is given by $F(z)$ where $F'(z) = -\text{Log}(1-z)$. If you recall from first year calculus, the primitive for $\ln(x)$ is $x\ln(x) - x$. We try something similar:

$$F(z) = (1-z)\text{Log}(1-z) - (1-z) + C$$

Does this function work? Let's double check:

$$F'(z) = (1-z)\frac{-1}{1-z} - \text{Log}(1-z) + 1 = -\text{Log}(1-z)$$

So this function is indeed a primitive for $-\text{Log}(1-z)$. We just need to find C to finish up. Well, $F(0) = -1 + C$. Also, $F(0) = \sum_{n=2}^{\infty} \frac{0^n}{n(n-1)} = 0$. As such, $C = 1$ and $F(z) = (1-z)\text{Log}(1-z) + z$. And since we have taken primitives from a series with radius of convergence $R = 1$, we have that this is valid if $|z| < 1$. In particular at $z = \frac{-1}{2}$. We find:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{2^n(n^2-n)} = F\left(-\frac{1}{2}\right) = \frac{3}{2}\text{Log}\left(\frac{3}{2}\right) - \frac{1}{2} = \frac{3}{2}\ln\left(\frac{3}{2}\right) - \frac{1}{2}$$

4.3 Integrating Around Removable Discontinuities

We've spent a great deal of time talking about power series, and developing their theory. What good does that do us? How can we use this theory? One way is to handle integrating around a removable discontinuity.

To begin, we show that removable discontinuities can be "removed" to give an analytic function. Recall that an isolated singularity z_0 is called removable if $\lim_{z \to z_0} f(z)$ exists.

Theorem 4.3.1

Suppose $f(z)$ is analytic on $D\setminus\{z_0\}$, has a removable discontinuity at $z_0 \in D$, and that $\lim_{z\to z_0} f(z) = L$. Then the function $\tilde{f}(z) = \begin{cases} f(z), & z \neq z_0 \\ L, & z = z_0 \end{cases}$ is analytic on D.

Proof. Since $\tilde{f}(z) = f(z)$ on $D \setminus \{z_0\}$, we know that $\tilde{f}$ is differentiable on $D \setminus \{z_0\}$. As such, we only need to show that $\tilde{f}$ is differentiable at z_0. However, this is not immediately accessible from the definition of the derivative. We instead consider a new function. Define:

$$k(z) = (z - z_0)\tilde{f}(z)$$

Since $\tilde{f}$ is differentiable on $D \setminus \{z_0\}$, so is $k(z)$. At z_0 we have:

$$k'(z_0) = \lim_{h\to 0} \frac{(z_0 + h - z_0)\tilde{f}(z_0 + h)}{h} = \lim_{h\to 0} \tilde{f}(z_0 + h) = \lim_{h\to 0} f(z_0 + h) = L$$

As such, $k(z)$ is differentiable at z_0 as well. Since k is analytic on D and $z_0 \in D$, k has a power series expansion valid on $B_r(z_0)$ for some $r > 0$. In particular:

$$k(z) = k(z_0) + k'(z_0)(z - z_0) + \sum_{n=2}^{\infty} \frac{k^{(n)}(z_0)}{n!}(z - z_0)^n = L(z - z_0) + \sum_{n=2}^{\infty} \frac{k^{(n)}(z_0)}{n!}(z - z_0)^n$$

Now, for $z \neq z_0$, $\tilde{f}(z) = \frac{k(z)}{z-z_0} = L + \sum_{n=2}^{\infty} \frac{k^{(n)}(z_0)}{n!}(z - z_0)^{n-1}$. However, note that when we evaluate this power series at z_0 we get $\tilde{f}(z_0) = L = L + \sum_{n=2}^{\infty} \frac{k^{(n)}(z_0)}{n!}(z_0 - z_0)^{n-1}$. So $\tilde{f}$ is given by this power series on $B_r(z_0)$!

However, we know that power series with positive radii of convergence are analytic. Since $\tilde{f}$ is described by a power series with positive radius of convergence on $B_r(z_0)$, $\tilde{f}$ is analytic on $B_r(z_0)$ and hence is differentiable at z_0, completing the proof. $\square$

How does this help us evaluate integrals though?

Theorem 4.3.2

Suppose $f(z)$ is analytic on a domain $D \setminus \{z_0\}$ and has a removable discontinuity at $z_0 \in D$. Suppose γ is a piecewise smooth closed curve in $D \setminus \{z_0\}$ such that the inside of γ is contained in D. Then:

$$\int_\gamma f(z)dz = 0$$

Proof. By theorem 4.3.1, there exists a function $\tilde{f}$ analytic on D such that $\tilde{f}(z) = f(z)$ for $z \neq z_0$. Now, since γ is contained in $D \setminus \{z_0\}$, $\tilde{f} = f$ on γ. As such:

$$\int_\gamma f(z)dz = \int_\gamma \tilde{f}(z)dz \overset{CIT}{=} 0$$

$\square$

Let's see this in action:

Example 4.3.1

Find $\int_{|z|=1} \frac{\sin(z)}{z} dz$.

Note that $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...$, and so for $z \neq 0$ we have:

$$\frac{\sin(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - ...$$

As such, $\lim_{z\to 0} \frac{\sin(z)}{z} = 1$. So the singularity is removable. By our previous theorem:

$$\int_{|z|=1} \frac{\sin(z)}{z} dz = 0$$

4.4 A Return to Poles

How do we compute an integral such as $\int_{|z|=4} \frac{z}{\sin(z)} dz$? This function has singularities $k\pi$ for $k \in \mathbb{Z}$, and so has three singularities inside the circle: $0, \pm\pi$. Now, as in our last example from the previous lecture, $z = 0$ is removable. But what about $\pm\pi$? Are they poles, or essential? How do we tell?

Recall, definition 3.3.3 tells us that z_0 is a pole of order n for $f(z)$ if we can write $f(z) = \frac{g(z)}{(z-z_0)^n}$ where $g(z)$ is analytic on a ball around z_0 and $g(z_0) \neq 0$. In particular, this tells us that $g(z) = (z - z_0)^n f(z)$ for $z \neq z_0$. I.e., $(z - z_0)^n f(z)$ has a removable singularity at $z = z_0$!

So, that means that $g(z) = \begin{cases} (z - z_0)^n f(z), & z \neq z_0 \\ \lim_{z\to z_0}(z - z_0)^n f(z), & z = z_0 \end{cases}$. As such, if γ is a circle centered at z_0 which encircles no other singularities (i.e., $f(z)$ is analytic on $D = B_r(z_0) \setminus \{z_0\}$ where γ is inside D), then:

$$\int_\gamma f(z)dz = \int_\gamma \frac{(z - z_0)^n f(z)}{(z - z_0)^n} dz \overset{CIF}{=} \frac{2\pi i}{(n-1)!} g^{(n-1)}(z_0)$$

That means we need to be able to compute $\lim_{z\to z_0}(z - z_0)^n f(z)$. This is a $0 \times \infty$ type of limit, which is precisely the setup for L'Hopital's rule.

However, to do so we need to discuss different types of zeroes of functions.

Definition 4.4.1: Zero of order n

Suppose $f(z)$ is analytic on a domain D and $z_0 \in D$. Then z_0 is a zero of order n if:

$$f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0$$

Zeroes of order 1 are called simple zeroes.

We now state and prove one version of L'Hopital's rule.

Theorem 4.4.1: L'Hopital's rule for $\frac{0}{0}$ forms

Suppose $f(z)$, $g(z)$ are analytic on a domain D, and $z_0 \in D$. Further, suppose that z_0 is a zero of order m for $f(z)$ and a zero of order k for $g(z)$. Then:

- If $m > k$, then $\lim_{z \to z_0} \frac{f(z)}{g(z)} = 0$.
- If $m < k$, then $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \infty$.
- If $m = k$, then $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)}$.

Proof. Since $f(z), g(z)$ are analytic on D and $z_0 \in D$, we then $f(z), g(z)$ have power series centered at z_0 with positive radius of convergence. In particular, there exists $R > 0$ such that on $B_R(z_0)$ we have:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$
$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^n$$

However, we know that z_0 is a zero of order m for f and a zero of order k for g. As such:

$$f(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$
$$g(z) = \sum_{n=k}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^n$$

We now look at the quotient, in our various cases.

Case 1. $m > k$

Suppose $m > k$. Then:

$$\frac{f(z)}{g(z)} = \frac{\frac{1}{(z-z_0)^k} \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n}{\frac{1}{(z-z_0)^k} \sum_{n=k}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^n} = \frac{\sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^{n-k}}{\sum_{n=k}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^{n-k}}$$

Notice that the denominator has a constant term of $\frac{g^{(k)}(z_0)}{k!}$, which is non-zero. In the numerator, the powers of $(z - z_0)$ that occur are $m - k$, $m - k + 1$, etc. Notice that these are all positive powers. As such:

$$\lim_{z \to z_0} \frac{f(z)}{(z - z_0)^k} = 0$$

$$\lim_{z \to z_0} \frac{g(z)}{(z - z_0)^k} = \frac{g^{(k)}(z_0)}{k!}$$

This gives that $\lim_{z \to z_0} \frac{f(z)}{g(z)} = 0$.

Case 2. $m < k$

Suppose $m < k$. Then:

$$\frac{f(z)}{g(z)} = \frac{\frac{1}{(z-z_0)^m} \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n}{\frac{1}{(z-z_0)^m} \sum_{n=k}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^n} = \frac{\sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^{n-m}}{\sum_{n=k}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^{n-m}}$$

Notice that the denominator has a no constant term. And the numerator has a constant term of $\frac{f^{(m)}(z_0)}{m!}$.

$$\lim_{z \to z_0} \frac{f(z)}{(z - z_0)^m} = \frac{f^{(m)}(z_0)}{m!}$$

$$\lim_{z \to z_0} \frac{g(z)}{(z - z_0)^m} = 0$$

This gives that $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \infty$.

Case 3. $m = k$

Performing the same argument again, this time we have that

$$\lim_{z \to z_0} \frac{f(z)}{(z - z_0)^m} = \frac{f^{(m)}(z_0)}{m!}$$

$$\lim_{z \to z_0} \frac{g(z)}{(z - z_0)^m} = \frac{g^{(m)}(z_0)}{m!}$$

And so:

$$\lim_{z \to z_0} \frac{\frac{f(z)}{(z-z_0)^m}}{\frac{g(z)}{(z-z_0)^m}} = \frac{\frac{f^{(m)}(z_0)}{m!}}{\frac{g^{(m)}(z_0)}{m!}} = \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)}$$

□

How does this help us?

Example 4.4.1

Find $\int_{|z|=4} \frac{z}{\sin(z)} dz$.

By the deformation theorem:

$$\int_{|z|=4} \frac{z}{\sin(z)} dz = \int_{|z|=1} \frac{z}{\sin(z)} dz + \int_{|z-\pi|=1} \frac{z}{\sin(z)} dz + \int_{|z+\pi|=1} \frac{z}{\sin(z)} dz$$

So how do we handle each of these integrals? Well, let's try to see if they're removable or poles.

For $z_0 = 0$, we have that $\lim_{z\to 0} \frac{z}{\sin(z)} = \frac{1}{\cos(0)} = 1$ by L'Hoptial. So $z_0 = 0$ is removable and the first integral is 0.

For $z_0 = \pi$, we see that $\lim_{z\to 0} \frac{z}{\sin(z)} = \infty$ since the denominator is approaching 0 but the numerator is not. Therefore, this is not removable.

Is it a pole of order 1? To see this, we check: $\lim_{z\to z_0} \frac{(z-\pi)z}{\sin(z)}$. Using L'Hopital's rule, we find that this is:

$$\lim_{z\to z_0} \frac{(z-\pi)z}{\sin(z)} = \lim_{z\to\pi} \frac{2z-\pi}{\cos(z)} = -\pi$$

So $\frac{z}{\sin(z)}$ has a pole of order 1, and that $g(z) = \frac{(z-\pi)z}{\sin(z)}$ has a removable discontinuity which can be made analytic by setting $g(\pi) = -\pi$. As such:

$$\int_{|z-\pi|=1} \frac{z}{\sin(z)} dz = 2\pi i g(\pi) = -2\pi^2 i$$

And for $z_0 = -\pi$, setting $g_2(z) = \frac{(z+\pi)z}{\sin(z)}$ gives:

$$\lim_{z\to 0} g_2(z) = \frac{-\pi}{-1} = \pi$$

And so:

$$\int_{|z-\pi|=1} \frac{z}{\sin(z)} dz = 2\pi i g_2(\pi) = 2\pi^2 i$$

All together, this gives us that:

$$\int_{|z|=4} \frac{z}{\sin(z)} dz = 0$$

This example shows us how to handle simple poles. What about poles of higher order? Well, we could use this argument. However, once we develop Laurent series, we can give a slightly more straightforward formula.

To end, let's talk about how to recognize the order of a pole intuitively. You will have a homework problem asking you to prove the following:

Theorem 4.4.2

Suppose $f(z)$ has a zero of order n at z_0 and $g(z)$ has a zero of order m at z_0. Then:

- If $n \geq m$, then $\frac{f(z)}{g(z)}$ has a removable discontinuity at z_0.
- If $n < m$, then $\frac{f(z)}{g(z)}$ has a pole of order $m - n$ at z_0.

4.5 Laurent Series

To be able to handle essential singularities, we're going to need to develop a new kind of series representation: the Laurent series for a function. This is similar to power series, except now we will allow negative powers of $(z - z_0)$. This will also make handling integrating around poles easier as well.

Definition 4.5.1: Laurent Series

A Laurent series centered at z_0 is a function $f(z)$ of the form:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}$$

The function is defined wherever both of these series exist, and is undefined whenever one diverges.

Before we get into how this idea is useful, let's start with an example:

Example 4.5.1

Suppose $f(z) = \frac{1}{1-z}$. We know that on $|z| < 1$ that $f(z) = \sum_{n=0}^{\infty} z^n$.

What about for $|z| > 1$? Well, we can rewrite:

$$f(z) = \frac{1}{z} \frac{1}{\frac{1}{z} - 1} = -\frac{1}{z} \frac{1}{1 - \frac{1}{z}}$$

Well, we know that $|z| > 1$, so $\left|\frac{1}{z}\right| < 1$. So, we can use the power series valid on $|z| < 1$ to get that:

$$\frac{1}{1 - \frac{1}{z}} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

All together, this gives us that:

$$\frac{1}{1 - z} = -\sum_{n=1}^{\infty} \frac{1}{z^n}$$

Alright, so let's tackle some theoretical concerns. In particular, we're interested in a few questions:

- Is there a radius of convergence type condition that let's us see when Laurent series converge?
- Are Laurent series analytic?
- Do analytic functions admit Laurent series?

Regarding the first question: we broke $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ into two sums. The sum with the positive powers:

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

is a power series. As such, it has a radius of convergence R_1 so that it converges on $B_{R_1}(z_0)$, and diverges when $|z-z_0| > R_1$.

To investigate the negative terms, we'll make a change of variables. Let $w = \frac{1}{z-z_0}$. Then we have:

$$\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n} = \sum_{n=1}^{\infty} a_{-n} w^n$$

which is a power series in w. As such, it has a radius of convergence R_2 so that it converges when $|w| < R_2$ and diverges when $|w| > R_2$. As such, this series converges when $|z-z_0| > \frac{1}{R_2}$ and diverges when $|z-z_0| < \frac{1}{R_2}$.

As such, there are two radii r_1 and r_2 such that the series converges when $r_1 < |z-z_0| < r_2$. Be careful: this does not automatically imply that $r_1 < r_2$. We need $r_1 < r_2$ for the series to converge anywhere, but we can easily find series such that $r_1 > r_2$, so that when the positive powers converge, the negative powers diverge.

Regarding analyticity:

Theorem 4.5.1

If $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ converges when $R_1 < |z-z_0| < R_2$ and $R_1 < R_2$, then:

$$f'(z) = \sum_{n=-\infty}^{\infty} n a_n (z-z_0)^{n-1}$$

In particular, this new series also converges when $R_1 < |z-z_0| < R_2$.

We won't be proving this. It's another theoretical result needing uniform convergence, like the fact that power series are analytic.

Much more interesting is that analytic functions admit Laurent series, and that the coefficients of the Laurent series are integrals!

Theorem 4.5.2

Suppose $f(z)$ is analytic on $D = \{z \in \mathbb{C} | R_1 < |z-z_0| < R_2\}$. Let $r \in (R_1, R_2)$. Then on D, $f(z)$ is given by the Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where $a_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz$.

Proof. Let $w \in D$, and choose $r_1, r_2 \in \mathbb{R}$ so that $R_1 < r_1 < |w - z_0| < r_2 < R_2$. By problem 3b on practice test 2, we have that:

$$f(w) = \frac{1}{2\pi i} \left(\int_{|z-z_0|=r_2} \frac{f(z)}{z-w} dz - \int_{|z-z_0|=r_1} \frac{f(z)}{z-w} dz \right)$$

Now, as in our proof of 4.2.1, we know that:

$$\int_{|z-z_0|=r_2} \frac{f(z)}{z-w} dz = \sum_{n=0}^{\infty} \left(\int_{|z-z_0|=r_2} \frac{f(z)}{(z-z_0)^{n+1}} dz \right) (w - z_0)^n$$

So we only need to handle the other integral. To do so, we'll rewrite:

$$\frac{f(z)}{z-w} = \frac{f(z)}{(z-z_0) - (w-z_0)} = \frac{1}{z-z_0} \frac{f(z)}{\left(1 - \frac{w-z_0}{z-z_0}\right)}$$

Now, since we're concerned with the integral around the circle $|z - z_0| = r_1$, we know that $|z - z_0| < |w - z_0|$ by our initial assumption. As such, $\left| \frac{w-z_0}{z-z_0} \right| > 1$. We may therefore use our Laurent series for $\frac{1}{1-z}$ on the annulus $|z| > 1$ to get:

$$\frac{f(z)}{z-w} = -\frac{1}{z-z_0} \sum_{n=1}^{\infty} \frac{f(z)(z-z_0)^n}{(w-z_0)^n}$$

Uniform convergence then gives us that:

$$\int_{|z-z_0|=r_1} \frac{f(z)}{z-w} dz = -\sum_{n=1}^{\infty} \left(\int_{|z-z_0|=r_1} f(z)(z-z_0)^{n-1} dz \right) \frac{1}{(w-z_0)^n}$$

This isn't in quite the form we want it in. So I'll make a change of variable on the index. Let $k = -n$. Then we have:

$$\int_{|z-z_0|=r_1} \frac{f(z)}{z-w} dz = -\sum_{k=-\infty}^{-1} \left(\int_{|z-z_0|=r_1} \frac{f(z)}{(z-z_0)^{k+1}} dz \right) (w - z_0)^k$$

Adding our two integrals together gives:

$$f(w) = \frac{1}{2\pi i}\left(\sum_{k=0}^{\infty}\left(\int_{|z-z_0|=r_2}\frac{f(z)}{(z-z_0)^{k+1}}dz\right)(w-z_0)^k + \sum_{k=-\infty}^{-1}\left(\int_{|z-z_0|=r_1}\frac{f(z)}{(z-z_0)^{k+1}}dz\right)(w-z_0)^k\right)$$

This is close to what we want, but not quite! Indeed, our final formula should involve integrals over $|z-z_0| = r$, not r_1 or r_2. However, since $\frac{f(z)}{(z-z_0)^{k+1}}$ is analytic on D, the deformation theorem tells us that

$$\left(\int_{|z-z_0|=r}\frac{f(z)}{(z-z_0)^{k+1}}dz\right) = \left(\int_{|z-z_0|=r_1}\frac{f(z)}{(z-z_0)^{k+1}}dz\right) = \left(\int_{|z-z_0|=r_2}\frac{f(z)}{(z-z_0)^{k+1}}dz\right)$$

This gives us the desired formula. □

So how does this help us to integrate? Well, what happens when we set $n = -1$? We get:

$$a_{-1} = \frac{1}{2\pi i}\int_{|z-z_0|=r} f(z)dz$$

So if we have a Laurent series, integrating becomes super simple! Let's see an example:

Example 4.5.2

Find $\int_{|z|=1} ze^{\frac{1}{z^2}}dz$.

With some effort, we can verify that this is neither removable nor a pole. This is an essential singularity. So we can't use CIT or CIF here. The only other options we have are to integrate by definition (terrible idea!) or to find a Laurent series. Mercifully, we know that $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$ for any $z \in \mathbb{C}$. For $z \neq 0$, $\frac{1}{z^2} \in \mathbb{C}$, and so:

$$ze^{\frac{1}{z^2}} = z\left(\sum_{n=0}^{\infty}\frac{1}{n!z^{2n}}\right) = \sum_{n=0}^{\infty}\frac{1}{n!z^{2n-1}} = z + \frac{1}{z} + \frac{1}{2!z^3} + \frac{1}{3!z^5} + ...$$

Now, to integrate this we only need to find that a_{-1} term. This is specifically the coefficient attached to the power z^{-1}. In this case, we have a z^{-1} term of $\frac{1}{z}$, which gives us $a_{-1} = 1$.

So, $\int_{|z|=1} ze^{\frac{1}{z^2}}dz = 2\pi i a_{-1} = 2\pi i$.

This is really useful. If you have a Laurent series valid on your curve, you can find the integral without any work.

4.6 Residues and The Residue Theorem

As we've seen, knowing that a_{-1} coefficient is really useful for integrating. The most common situation is where you are integrating over a punctured domain: a domain with one point removed. In this case, you know that your function $f(z)$ has a Laurent series at z_0 valid on $0 < |z-z_0| < R$ for some R. This comes up so often that we give a_{-1} a name in this case:

Definition 4.6.1: Residue

Suppose $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ converges on the annulus $\{z \in \mathbb{C} | 0 < |z-z_0| < R\}$ for some $R > 0$. Then the **residue** of $f(z)$ at z_0 is:

$$\text{Res}(f; z_0) = a_{-1}$$

Warning 4.6.1

In a Laurent series, a_{-1} is called the residue ONLY if the inner radius of the annulus is 0. Otherwise it is not the residue.
In particular, you can easily run into situations where the function both has a Laurent series defined on $0 < |z - z_0| < R_1$ and on $r_2 < |z - z_0| < R_2$ with different a_{-1} coefficients occuring. The reside only corresponds to $0 < |z - z_0| < R$. However, if you are integrating in the region $r_2 < |z - z_0| < R_2$, you will need that a_{-1} coefficient that is valid in **that region**.

So, for example:

Example 4.6.1

From our earlier example, $\text{Res}(ze^{\frac{1}{z^2}}; 0) = 1$.

Example 4.6.2

Find $\text{Res}(\frac{1}{1-z}; 0)$.

Well, we know that $\frac{1}{1-z}$ is analytic inside $B_1(0)$. As such, it has a power series. A power series is just a Laurent series with $a_n = 0$ if $n < 0$. So $a_{-1} = 0$ and $\text{Res}(\frac{1}{1-z}; 0) = 0$.

This same argument proves:

Theorem 4.6.1

If $f(z)$ is analytic on $B_R(z_0)$ or has a removable singularity at z_0, then $\text{Res}(f; z_0) = 0$.

Proof. We've already discussed what happens when $f(z)$ is analytic on $B_R(z_0)$.

If f has a removable singularity at z_0, then we have previously shown (in the proof of 4.3.1 that $f(z)$ is given by a power series on $B_R(z_0) \setminus \{z_0\}$ for some $R > 0$. This is an annulus whose inner radius is 0, and so the a_{-1} term of this power series is the residue. As such, the residue is 0. □

We've used this principle to compute one integral (our first example with $ze^{\frac{1}{z^2}}$). Does this apply more generally? It turns out it's fairly flexible:

Theorem 4.6.2: The Residue Theorem

Let D be a domain and $z_1, \ldots, z_n \in D$. Suppose $f(z)$ is analytic on $D \setminus \{z_1, \ldots, z_n\}$. Let γ be a piecewise smooth, positively oriented, simple closed curve in D such that the inside of γ is in D and $z_1, \ldots, z_n$ are inside γ. Then:

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} 2\pi i \text{Res}(f; z_k)$$

What this really says is that the integral is $2\pi i$ times the sum of the residues at the function's singularities. And in particular, the singularities **inside** the curve.

Proof. With the conditions on γ and f give, the deformation theorem applies to give that:

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{|z-z_k|=\frac{r_k}{2}} f(z)dz$$

where $r_k > 0$ such that $B_{r_k}(z_k) \in D \setminus \{z_1, \ldots, z_k\}$. However, since $f(z)$ is analytic on $B_{r_k}(z_k) \setminus \{z_k\}$, we know that $|z - z_k| = \frac{r_k}{2}$ is contained in an annulus of radius 0 on which $f(z)$ is analytic. As such:

$$\int_{|z-z_k|=\frac{r_k}{2}} f(z)dz = 2\pi i \text{Res}(f; z_k)$$

giving us the desired result.

□

All this business of integrating using Laurent series and residues takes a bit of care.

Example 4.6.3

We know, by using the Cauchy Integral Formula that $\int_{|z|=2} \frac{1}{1-z} dz = -2\pi i$. So what's wrong with the following argument?

> We know that $\int_{|z|=2} \frac{1}{1-z} dz = a_{-1}$ in a Laurent series for $\frac{1}{1-z}$. However, $a_{-1} = \text{Res}(f; 0)$. Since $\frac{1}{1-z}$ is analytic on $B_1(0)$, we have that $\text{Res}(\frac{1}{1-z}; 0) = 0$ and so $\int_{|z|=2} \frac{1}{1-z} dz = 0$.

The key thing here is that we're using the wrong Laurent series! We want a_{-1} from a Laurent series of $\frac{1}{1-z}$ which is valid on $|z| = 2$. However, we know that the power series for $\frac{1}{1-z}$ is only valid on $|z| < 1$, which does not contain the curve. The correct Laurent series to use is:

$$\frac{1}{1-z} = -\sum_{n=1}^{\infty} \frac{1}{z^n}$$

which gives $a_{-1} = -1$, agreeing with our calculating using CIF.

Another way to look at this is: we used the wrong residue! We tried to say that $\int_{|z|=2} \frac{1}{1-z} dz = 2\pi i \text{Res}(\frac{1}{1-z}; 0)$. But that's not what the residue theorems says to do. We need $\text{Res}(\frac{1}{1-z}; 1)$. Since

$\frac{1}{1-z} = -\frac{1}{z-1}$ is already written as a Laurent series centered at $z_0 = 1$, we can quickly read off the residue as -1.

This example brings up an important warning:

Warning 4.6.2

Use the right Laurent series! Make sure that when you're using Laurent series to integrate, you're using a Laurent series that is valid on an annulus containing your curve!

And when trying to use the Residue Theorem, plug the right points in! The whole idea is that you look at the residues at the singularities!

The Residue Theorem gives us an overarching theory for integrating functions which are analytic except for some number of isolated singularities. However, this simplicity is really just an organizational tool. We've moved the hard part from knowing a bunch of different theorems to knowing how to calculate residues. We've handled removable singularities. That leaves poles and essential singularities.

4.6.1 Residues at Poles

Let's start by talking about how to recognize poles from their Laurent series.

Suppose $f(z)$ has a pole of order $n > 0$ at z_0. Then we know, from the definition of a pole, that $f(z) = \frac{g(z)}{(z-z_0)^n}$ where $g(z_0) \neq 0$ and $g(z)$ is analytic. Since $g(z)$ is analytic, it has a power series centered at z_0. So we can write:

$$f(z) = \frac{\sum_{k=0}^{\infty} b_k (z-z_0)^k}{(z-z_0)^n} = \frac{b_0}{(z-z_0)^n} + \frac{b_1}{(z-z_0)^{n-1}} + \dots$$

Notice that this is a Laurent series with $a_k = b_{k+n}$, and so $a_k = 0$ for $k < -n$. In particular, there are only finitely many terms with negative powers! And since $b_0 = g(z_0) \neq 0$, we know that z^{-n} is the lowest power that occurs in this series.

Okay, so this tells us how to recognize a pole of order n. But how do we compute residues? This turns out to be very painful.

Theorem 4.6.3

Suppose $f(z)$ has a pole of order n at z_0. Then:

$$\operatorname{Res}(f; z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{d\, z^{n-1}} (z-z_0)^n f(z)$$

Proof. To start, let's look at $(z-z_0)^n f(z)$. We write $f(z) = \sum_{k=-n}^{\infty} a_k (z-z_0)^k$. Then:

$$(z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) + a_{-n+2}(z-z_0)^2 + \dots$$

We need to isolate a_{-1}. Let's see what differentiating does:

$$\frac{d}{d\,z}(z-z_0)^n f(z) = a_{-n+1} + 2a_{-n+2}(z-z_0) + 3a_{-n+3}(z-z_0)^2 + \dots$$
$$\frac{d^2}{d\,z^2}(z-z_0)^n f(z) = 2a_{-n+2} + (3)(2)a_{-n+3}(z-z_0) + \dots$$

Inductively, we can show that:

$$\frac{d^{n-1}}{d\,z^{n-1}}(z-z_0)^n f(z) = (n-1)!a_{-1} + n!a_0(z-z_0) + \dots$$

And so $\lim_{z\to z_0} \frac{d^{n-1}}{d\,z^{n-1}}(z-z_0)^n f(z) = (n-1)!a_{-1} = (n-1)!\text{Res}(f;z_0)$. (Why can't we just substitute in $z = z_0$ at this point?)

Dividing by $(n-1)!$ gives the desired result.

□

Let's see this in practice. I apologize in advance.

Example 4.6.4

Find $\int_{|z|=1} \frac{1}{\sin^2(z)} dz$.

Well, $\frac{1}{\sin^2(z)}$ is analytic on $\{z \in \mathbb{Z} | 0 < |z| < \pi\}$, so the Residue Theorem applies to give:

$$\int_{|z|=1} \frac{1}{\sin^2(z)} dz = 2\pi i \text{Res}\left(\frac{1}{\sin^2(z)}; 0\right)$$

so we only need to find this residue. We need to identify what type of singularity this is first. We have already seen that $\sin(z)$ has a simple zero at $z_0 = 0$, so $\frac{1}{\sin(z)}$ has a simple pole there. We expect that $\frac{1}{\sin^2(z)}$ has a double pole there. Indeed, L'Hopital's rule quickly gives us that:

$$\lim_{z\to 0} \frac{z^2}{\sin^2(z)} = 1$$

so this is a double pole. Unfortunately, this is not the limit we need to compute to find the residue. This limit only tells us what type of pole we have. Instead, we have:

$$\text{Res}\left(\frac{1}{\sin^2(z)}; 0\right) = \lim_{z\to 0} \frac{d}{d\,z} \frac{z^2}{\sin^2(z)} = \lim_{z\to 0} \frac{2z\sin(z) - 2z^2\cos(z)}{\sin^3(z)}$$

From here, we use L'Hopital's rule **twice** to get:

$$\text{Res}\left(\frac{1}{\sin^2(z)}; 0\right) = \lim_{z\to 0} \frac{4z\sin(z) + 2(z^2+1)\cos(z) + z\sin(z) - 2\cos(z)}{3\cos(2z)} = 0$$

And so our integral is 0 as well.

Generally, for a pole of order n you should expect to use L'Hopital's rule n times to compute this limit.

4.6.2 Residues at Essential Singularities

If you're trying to integrate around an essential singularity, you have no choice but to find a Laurent series. I'd like to say there's some trick, but there really isn't. Generally, this means you're going to be using power series you already know, such as our earlier example with $ze^{\frac{1}{z^2}}$.

It is worth mentioning that we can recognize essential singularities by the function's Laurent series.

Theorem 4.6.4

If $f(z)$ is analytic on $B_R(z_0) \setminus \{z_0\}$ and $f(z)$ has Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$, then z_0 is essential if and only if for any $m \in \mathbb{Z}$, there exists $n \in \mathbb{Z}$ with $n < m$ and $a_n \neq 0$.

So, if you have a Laurent series that extends infinitely in the negative direction and has inner radius of convergence 0, the singularity is essential.

Proof. We proceed by showing the contrapositive.

If the condition that for any $m \in \mathbb{Z}$ there exists $n < m$ with $a_n \neq 0$ is false, then there exists some $m \in \mathbb{Z}$ such that if $n < m$, then $a_n = 0$. If $m < 0$, we have seen from the previous subsection that z_0 is a pole of order m. And if $m \geq 0$, then z_0 is removable. □

4.7 Contour Integration - Rational Functions

Now that we have a handle on how to integrate complex functions along closed curves, let's see one use for it. A surprising use is that with a bit of creativity we can compute improper integrals:

$$\int_{-\infty}^{\infty} f(x)dx$$

for a surprising array of functions. Today, we'll see how to do this for rational functions.

To begin, let's remind ourselves what this improper integral means:

Definition 4.7.1: Improper Integrals

Suppose $f(x)$ is continuous on $\mathbb{R}$. Then the improper integral $\int_{-\infty}^{\infty} f(x)dx$ is defined to be:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to\infty} \int_a^r f(x)dx + \lim_{s\to-\infty} \int_s^a f(x)dx$$

for any $a \in \mathbb{R}$, and whenever both limits exist. If both limits converge, we say the improper integral converges.

We won't be working with two limits. For our purposes, we need one limit: $\lim_{r\to\infty} \int_{-r}^{r} f(x)dx$. Thankfully, if the integral converges then:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to\infty} \int_{-r}^{r} f(x)dx$$

This limit actually has a special name:

Definition 4.7.2: Principal Value of an Improper Integral

The principal value P.V. $\int_{-\infty}^{\infty} f(x)dx$ is $\lim_{r\to\infty} \int_{-r}^{r} f(x)dx$.

So, to compute the improper integrals we're interested in, we're going to first show the integral exists and then calculate the principal value.

We're interested in rational functions. So we'll handle the general case:

Theorem 4.7.1

Suppose $p(z), q(z)$ are polynomials. Then:

- If $q(x)$ has roots in $\mathbb{R}$ which are not roots of $p(x)$ then $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}dx$ diverges.
- If $\deg p(x) \geq \deg q(x) - 1$, then $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}dx$ diverges.
- If $\deg p(x) \leq \deg q(x) - 2$ and $q(x) \neq 0$ for $x \in \mathbb{R}$, then $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}dx$ converges,

Proof. In the first two cases, comparing the function to $\frac{1}{x}$ gives the desired divergence.

For the final case, suppose $p(x)$ has real roots $r_1 < \cdots < r_n$. Since $\frac{p(x)}{q(x)}$ is continuous on $[r_1, r_n]$, then $\int_{r_1}^{r_n} \frac{p(x)}{q(x)}dx$ exists. So we only need to handle the two tails.

By the intermediate value theorem, $\frac{p(x)}{q(x)}$ is either always positive or always negative on each of the intervals $(-\infty, r_1]$ and $[r_n, \infty)$. So we can apply the limit comparison test. In particular, $\lim_{x\to\pm\infty} \frac{\frac{p(x)}{q(x)}}{\frac{1}{x^2+1}}$ exists.

And since $\int_{-\infty}^{\infty} \frac{1}{x^2+1}dx = \pi$, the limit comparison test gives us that $\int_{-\infty}^{r_1} \frac{p(x)}{q(x)}dx$ and $\int_{r_n}^{\infty} \frac{p(x)}{q(x)}dx$ also exist. $\square$

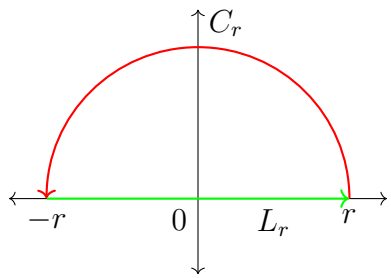
So now we know that the integrals we care about exist, how do we actually compute the integral?

Example 4.7.1

Find $\int_{-\infty}^{\infty} \frac{x+1}{x^4+2x^2+1}dx$.

By our theorem, this integral exists. We need to somehow turn this real integral into a complex one.

We're interested in $\int_{-r}^{r} \frac{x+1}{x^4+2x^2+1}dx$. To get a closed curve in $\mathbb{C}$, we look at:



If $f(z) = \frac{z+1}{z^4+2z^2+1}$, then the integral we care about is:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to\infty} \int_{-r}^{r} f(x)dx = \lim_{r\to\infty} \int_{L_r} f(z)dz$$

To compute this, we'll proceed in three steps:

1. Show that $\lim_{r\to\infty} \int_{C_r} f(z)dz = 0$.
2. Show that $\int_{-\infty}^{\infty} f(z)dz = \lim_{r\to\infty} \int_{L_r+C_r} f(z)dz$.
3. Compute $\lim_{r\to\infty} \int_{L_r+C_r} f(z)dz$.

Step 1. To do this, we'll use M-L estimation (3.2.7). Since C_r is a semicircle of radius r, it has length $L = \pi r$.

So we need to estimate the maximum of $|f(z)|$ on C_r. Since $f(z) = \frac{z+1}{z^4+2z^2+1}$, we can get an upper bound for $|f(z)|$ by getting an upper bound for $|z^2|$ and a lower bound for $|z^4 + 2z^2 + 1|$.

By the triangule inequality, $|z + 1| \leq |z| + 1$. Since z is on C_r, $|z| = r$. So $|z + 1| \leq r + 1$.

Also, by a variation of the triangle inequality (specifically that $|a - b| \geq ||a| - |b|)$, we see that $|z^4 + 2z^2 + 1| \geq r^4 - 2z^2 - 1$ for r large enough (bigger than 2 should be enough.)

Putting these two bounds together, we see that $|f(z)| \leq \frac{r+1}{r^4-2r^2-1}$. And so, M-L esitmation gives us that:

$$\left| \int_{C_r} f(z)dz \right| \leq \frac{\pi r(r+1)}{r^4 - 2r^2 - 1}$$

.

So, in the limit:

$$0 \leq \lim_{r\to\infty} \left| \int_{C_r} f(z)dz \right| \leq \lim_{r\to\infty} \frac{\pi r(r+1)}{r^4 - 2r^2 - 1} = 0$$

And so $\lim_{r\to\infty} \int_{C_r} f(z)dz = 0$.

Step 2: We know that:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to\infty} \int_{L_r} f(z)dz$$

But since $\lim_{r\to\infty} \int_{C_r} f(z)dz = 0$, we have that:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to\infty} \int_{L_r} f(z)dz + \lim_{r\to\infty} \int_{C_r} f(z)dz = \lim_{r\to\infty} \int_{L_r+C_r} f(z)dz$$

which is what we wanted to show.

Step 3: We compute this integral using the residue theorem. Since $z^4 + 2z^2 + 1 = (z^2+1)^2$, we see that $f(z)$ has two double poles: $\pm i$. Of these, only i is inside the curve (when $r > 1$). As such, when $r > 1$ the residue theorem gives:

$$\int_{L_r+C_R} f(z)dz = 2\pi i \text{Res}\left(\frac{z+1}{z^4+2z^2+1}; i\right)$$

Since this is a double pole, we compute:

$$\begin{aligned}
\text{Res}\left(\frac{z+1}{z^4+2z^2+1}; i\right) &= \lim_{z\to i} \frac{d}{d\,z} \frac{(z-i)^2(z+1)}{(z^2+1)^2} \\
&= \lim_{z\to i} \frac{d}{d\,z} \frac{z+1}{(z+i)^2} \\
&= \lim_{z\to i} \frac{(z+i)^2 - 2(z+i)(z+i)}{(z+i)^4} \\
&= \frac{(2i)^2 - 2(2i)(i+1)}{(2i)^4} \\
&= \frac{-4+4-4i}{16} \\
&= \frac{-i}{4}
\end{aligned}$$

And so, all together:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to\infty} \int_{L_r+C_r} f(z)dz = 2\pi i \frac{-i}{4} = \frac{\pi}{2}$$

For other rational functions, the procedure is identical.

Appendices

A | Technical Proofs

A.1 The Extreme Value Theorem

To prove the Fundamental Theorem of Algebra, we used some technical results. In this subsection, we provide a proof. To begin, we need a couple of definitions.

Definition A.1.1: Closed Set

A set $K \subset \mathbb{C}$ is closed if $K^c = \{z \in \mathbb{C} | z \notin K\}$ is open.

Example A.1.1: Closed Balls

The closed ball $K = \{z \in \mathbb{C} || z - z_0| \leq R\}$ is closed.

Consider $w \notin K$. Then $|w - z_0| > R$. Let $\varepsilon > 0$ such that $R + \varepsilon < |w - z_0|$. We claim that $B_\varepsilon(w) \subset K^c$. To see this, let $z \in B_\varepsilon(w)$. Then:

$$|z - z_0| = |(w - z) - (w - z_0)| \geq ||w - z| - |w - z_0|| = |w - z_0| - \varepsilon > R$$

So $z \notin K$. As such, $B_\varepsilon(w) \subset K^c$.

Therefore, we have shown that for any $w \in K^c$, there is a ball $B_r(w)$ which is contained in K^c. So K^c is open, and therefore K is closed.

Definition A.1.2: Cauchy Sequence

A sequence $z_1, \ldots, z_n, \ldots$ is called Cauchy if for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that if $n, m > N$, then $|z_n - z_m| < \varepsilon$.

It is a neat fact, derived from the fact that $\mathbb{C}$ is topologically the same as $\mathbb{R}^2$, that $\mathbb{C}$ is complete: every Cauchy sequence in $\mathbb{C}$ converges.

Theorem A.1.1: Extreme Value Theorem

Suppose $f(z)$ is continuous on $\mathbb{C}$. Let $K \subset \mathbb{C}$ be closed and bounded. Then there exists $M \in \mathbb{R}$ such that $|f(z)| \leq M$ for all $z \in K$. I.e., $f(z)$ is bounded on K.

Proof. Suppose $f(z)$ is not bounded on K. I.e., for any $N \in \mathbb{C}$, there exists $z \in K$ such that $|f(z)| \geq N$.

Define $K_n = \{z \in \mathbb{C} || f(z)| > 2^n\}$. We know that K_n is non-empty for each n. For each n, choose $z_n \in K_n$.

Since K is bounded, there exists some $a < b$ and $c < d$ so that $K \subset D = \{z \in \mathbb{C} | a \leq \mathrm{Re}(z) \leq b, c \leq \mathrm{Im}(z) \leq d\}$.

Now, there are an infinite number of z_n. (Since for any $z \in \mathbb{C}$ there exists m with $|f(z)| < 2^m$, so for any n there exists m such that $z_n \neq z_j$ for all $j > m$.)

We cut D into four regions:

$$\{z \in \mathbb{C} | a \leq \mathrm{Re}(z) \leq \frac{a+b}{2}, c \leq \mathrm{Im}(z) \leq \frac{c+d}{2}\}$$

$$\{z \in \mathbb{C} | a \leq \mathrm{Re}(z) \leq \frac{a+b}{2}, \frac{c+d}{2} \leq \mathrm{Im}(z) \leq d\}$$

$$\{z \in \mathbb{C} | \frac{a+b}{2} \leq \mathrm{Re}(z) \leq b, c \leq \mathrm{Im}(z) \leq \frac{c+d}{2}\}$$

$$\{z \in \mathbb{C} | \frac{a+b}{2} \leq \mathrm{Re}(z) \leq b, \frac{c+d}{2} \leq \mathrm{Im}(z) \leq d\}$$

Since there are an infinite number of z_n and only four regions, one of these four rectangles contains an infinite number of the z_n. Set D_1 to be that region.

Repeat this procedure with D_1 to find another set $D_2 \subset D_1$ containing an infinite number of the z_n. Repeat to find D_j such that: $D_j \subset D_{j-1}$ and D_j contains an infinite number of these z_n.

From each D_j, choose one of the z_n in that $D_j \cap K_j$, which we will call w_j. This intersection is non-empty since D_j contains an infinite number of the z_n, and only finitely many of that z_n are not in K_j.

I claim that the w_j converge. To prove this, we will need to use a fact about $\mathbb{C}$: $\mathbb{C}$ is a complete metric space. This means that every Cauchy sequence in $\mathbb{C}$ converges. So, if we can prove that the sequence $w_1, w_2, \ldots$ is Cauchy, then we know it converges.

This turns out to be easy. Let $\varepsilon > 0$. In a rectangle with side lengths r, s, the greatest distance between any two points in the rectangle is given by considering the distance between two opposite vertices. This is $\sqrt{r^2 + s^2}$. By our construction, the rectangle D_j has side lengths $2^{-j}(b-a)$ and $2^{-j}(d-c)$. Therefore, choose N such that $2^{-N}\sqrt{(b-a)^2 + (d-c)^2} < \varepsilon$.

Then for any $n, m > N$, we have that $w_n \in D_n \subset D_N$ and $w_m \in D_m \subset D_N$. As such:

$$|w_n - w_m| \leq 2^{-N}\sqrt{(b-a)^2 + (d-c)^2} < \varepsilon$$

So the sequence is Cauchy, and hence converges to some w. Furthermore, since K is closed, we know that $w \in K$.

To finish up, we use that $f(z)$ is continuous. This tells us that:

$$|f(w)| = |f(\lim_{j\to\infty} w_j)| = |\lim_{j\to\infty} f(w_j)| = \lim_{j\to\infty} |f(w_j)|$$

However, we know that for any $N \in \mathbb{N}$, if $j > N$ then $w_j \in D_N \cap K_N$, and so $|f(w_j)| > 2^N$. As such, $|f(w)| = \lim_{j\to\infty} |f(w_j)| = \infty$. However, this is not possible since f is continuous at w. Contradiction.

Therefore, f is bounded on K.

$\square$

A.2 Series

Theorem A.2.1

Absolutely convergent series converge.

Proof. Suppose $\sum_{n=k}^{\infty} a_n$ converges absolutely. To show that it converges, we need to show that $(S_n)_{n=k}^{\infty}$ converges. As mentioned in subsection A.1 of these appendices, this is equivalent to showing that the sequence is Cauchy.

I.e., we need to show that we can force $|S_n - S_m| = |\sum_{j=m+1}^{n} a_j|$ to be very small.

Let $\varepsilon > 0$. Since $\sum_{n=k}^{\infty} |a_n|$ converges, there exists $N \in \mathbb{N}$ such that $n > m > N$ implies that $|T_n - T_m| < \varepsilon$ where $T_n = \sum_{j=k}^{n} |a_n|$.

Now, $|T_n - T_m| = |\sum_{j=m+1}^{n} |a_j|| = \sum_{j=m+1}^{n} |a_j|$.

By the triangle inequality:

$$|\sum_{j=m+1}^{n} a_n| \leq \sum_{j=m+1}^{n} |a_j|$$

And so $|S_n - S_m| \leq |T_n - T_m|$. Therefore, if $n, m > N$, then $|S_n - S_m| \leq |T_n - T_m| < \varepsilon$. This proves that the sequence $(S_n)_{n=k}^{\infty}$ is Cauchy, and therefore converges as well. □

For some of our theorem on analytic functions, we need to discuss the notion of uniformly convergent series.

Definition A.2.1: Uniform Convergence

Suppose $(f_n)_{n=k}^{\infty}$ is a sequence of functions, defined on some set S. We say that f_n converges uniformly to f on S if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } n > N \implies |f_n(z) - f(z)| < \varepsilon \forall z \in S$$

A uniformly convergent series is a series whose partial sums converge uniformly.

Notice the order of the quantifiers here. This is saying that N doesn't depend on z. So not only does the series converge to f at each point, but the series converges at roughly the same speed. More precisely, there's some lower bound to how fast the series converges at each point.

Why is this important? Well, it let's me get away with swapping some integrals and limits. In particular, it lets us do the following:

Theorem A.2.2

Suppose f_n are all continuous on the piecewise smooth curve γ and $f_n \to f$ uniformly. Then $\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz$.

Proof. Since $\left|\int_\gamma f_n(z)dz - \int_\gamma f(z)dz\right| \leq \int_\gamma |f_n(z) - f(z)|dz$, we prove that:

$$\lim_{n\to\infty} \int_\gamma |f_n(z) - f(z)|dz = 0$$

Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $|f_n(z) - f(z)| \leq \frac{\varepsilon}{L(\gamma)}$, where $L(\gamma)$ is the length of γ.

Now, by M-L estimation, $\int_\gamma |f_n(z) - f(z)|dz \leq \frac{\varepsilon}{L(\gamma)} L(\gamma) = \varepsilon$.

So, we have shown that $\forall \varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $n > N$ implies $\int_\gamma |f_n(z) - f(z)|dz = \left|\int_\gamma |f_n(z) - f(z)|dz - 0\right| < \varepsilon$. As such, $\lim_{n\to\infty} \int_\gamma |f_n(z) - f(z)|dz = 0$.

By the squeeze theorem, $\lim_{n\to\infty} \left|\int_\gamma f_n(z)dz - \int_\gamma f(z)dz\right| = 0$, and so $\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz$. □

All that remains for our purposes is to show that $\sum_{n=0}^\infty z^n$ converges uniformly on $|z| \leq R$ for any $R < 1$.

Theorem A.2.3

Let $0 \leq R < 1$. Then $\sum_{n=0}^\infty z^n$ converges uniformly to $\frac{1}{1-z}$ on $|z| \leq R$.

Proof. Let $D = \{z \in \mathbb{C} | |z| \leq R\}$. Let $S_n(z) = \sum_{j=0}^n z^j$. Recall that $S_n(z) = \frac{1-z^{n+1}}{1-z}$. So, to show that the sum converges uniformly, we need to show that $\frac{1-z^{n+1}}{1-z}$ converges uniformly to $\frac{1}{1-z}$.

Notice that $|S_n(z) - \frac{1}{1-z}| = \frac{z^{n+1}}{1-z}$. Now, since $|z| \leq R < 1$, we have that $|1 - z| \geq 1 - R > 0$. As such, $\frac{z^{n+1}}{1-z} \leq \frac{R^{n+1}}{1-R}$.

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} \frac{R^{n+1}}{1-R} = 0$, there exists $N \in \mathbb{N}$ such that if $n > N$, then $\frac{R^{n+1}}{1-R} < \varepsilon$.

However, this also gives that $|S_n(z) - \frac{1}{1-z}| \leq \frac{R^{n+1}}{1-R} < \varepsilon$ for all $z \in D$. As such, the series converges uniformly to $\frac{1}{1-z}$ on D. □

In actuality, we need that $\sum_{n=0}^\infty f(z)\frac{(w-z_0)^n}{(z-z_0)^{n+1}}$ where $|w - z_0| < |z - z_0|$ converges uniformly to $\frac{f(z)}{z-w}$ on a simple, closed curve γ. Since $f(z)$ is bounded on the curve, this is a simple modification of the above argument.

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